Information Theoretic Constraints Breed New Combinatorial Structures: Entropy Functions on Two-Dimensional Faces of Polymatroidal Region of Degree Four

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Entropy functions

Entropy function

Let $N_n = \{1, 2, \ldots, n\}$. For a discrete random vector $\mathbf{X} = (X_i, i \in N_n)$, the entropy function of $\mathbf{X}$ is a set function $\mathbf{h} : 2^{N_n} \to \mathbb{R}$ defined by

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Entropy region Γ_n^*

$$\Gamma_n^* \triangleq \{\mathbf{h} \in \mathcal{H}_n : \exists\ \mathbf{X},\ \mathbf{h} \text{ is the entropy function of } \mathbf{X}\}$$

Shannon outer bound Γ_n^*

Shannon-type inequalities

For any $A, B \subseteq N_n$,

$$H(X_A) \geq 0,$$
$$H(X_A) \leq H(X_B) \quad \text{if } A \subseteq B,$$
$$H(X_A) + H(X_B) \geq H(X_{A\cap B}) + H(X_{A\cup B}).$$

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Polymatroidal region Γ_n

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Non-Shannon-type Information inequalities

(Zhang-Yeung inequality, 1998) Given random variables X_1, X_2, X_3 and X_4,

$$2I(X_3; X_4) \leq I(X_1; X_2) + I(X_1; X_3, X_4) + 3I(X_3; X_4|X_1) + I(X_3; X_4|X_2).$$

Faces of a polyhdral cone

- Let $C \subseteq \mathbb{R}^d$ be a full-dimensional polyhedral cone. For a hyperplane P containing O in $\mathbb{R}^d$, if $C \subseteq P^+$, where P^+ is one of the two halfspaces corresponding to P,

$$F \triangleq C \cap P$$

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  - F is an extreme ray of C if $\dim F = 1$.

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  - H-representation: each face F can be written as the intersection of the facets containing F.
  - V-representation: each face F can be written as the convex combination of the extreme rays F contains.

Faces of Γ_n: facets

Elemental inequalities

$$\mathbf{h}(N_n) \geq \mathbf{h}(N_n \setminus \{i\}) \quad i \in N_n;$$

$$\mathbf{h}(K \cup i) + \mathbf{h}(K \cup j) \geq \mathbf{h}(K) + \mathbf{h}(K \cup ij) \quad i < j, i, j \in N_n, K \subseteq N_n \setminus \{i, j\}$$

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- Each facet $F = \Gamma_n \cap P$ one-to-one corresponds to a unique P which is the hyperplane by setting an elemental inequality by equality
- There are totally $n + \binom{n}{2} 2^{n-2}$ elemental inequality, and so $n + \binom{n}{2} 2^{n-2}$ facets of Γ_n

Faces of Γ_n: extreme rays

Obtain extreme rays by facets

Extreme rays of Γ_n can be otained from the facets by the software lrs for small number of n.

- for $n = 2$, there exist 3 extreme rays, while there are 3 facets
- for $n = 3$, there exist 8 extreme rays, while there are 9 facets
- for $n = 4$, there exist 41 extreme rays, while there are 28 facets
- for $n = 5$, there exist over 10^6 extreme rays, while there are 85 facets

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Extreme ray representation

$$E = \{a\mathbf{r} : a \geq 0, \}$$

where $\mathbf{r}$ is the minimal integer polymatroid on the ray.

Inequalities characterizations are speceial cases of face characterizations

Constrained information inequalities

For a set C of constraints of equalities obtained by setting the Shannon-type inequalities be equalities,

- $F = \Gamma_n \cap C$ is a face of Γ_n, and
- the constrained information inequalities under C determines an outer bound of the entropy functions on F

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Unconstrained information inequalities

For unconstrained information inequalities, we take the improper face $F = \Gamma_n$.

Entropy functions on faces of Γ_3: extreme rays

Extreme rays of Γ_3

8 extreme rays in 4 types are in the form

$$E_M = \{a\mathbf{r}_M : a \geq 0, \}$$

where M are

- $U^i_{1,1}$, $i \in N_3$;
- $U^\alpha_{1,2}, \alpha \subseteq N_3, |\alpha| = 2$;
- $U_{1,3}$;
- $U_{2,3}$

and $U^\alpha_{k,m}$ is the matroid on N_3 with rank function $\mathbf{r}(A) = \min\{|A \cap \alpha|, k\}, A \subseteq N_3$, and $\alpha = N_3$ when it is omitted.

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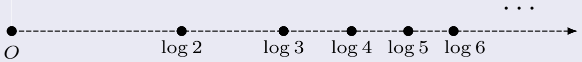
Entropy functions on extreme rays

- The first 7 extreme rays in 3 types are all entropic.
- $E^*_{U_{2,3}} = E_{U_{2,3}} \cap \Gamma^*_n = \{a\mathbf{r}_{U_{2,3}} : a \geq 0, a = \log k \text{ for some positive } k \in \mathbb{Z}\}$.[a]



[a] Zhen Zhang and Raymond W Yeung. "A non-Shannon-type conditional inequality of information quantities". In: *IEEE Transactions on Information Theory* 43.6 (1997), pp. 1982–1986.

Entropy functions on faces of Γ_3: 2-dim faces

- $F = (E_1, E_2) = \{a\mathbf{r}_1 + b\mathbf{r}_2 : a, b \geq 0\}$.
- Only two types of faces containing $U_{2,3}$ need to be further characterized: $(U_{2,3}, U_{1,2}^{12})$ and $(U_{2,3}, U_{1,1}^{1})$, which has been done by Matúš[1], and Chen and Yeung[2], respectively.

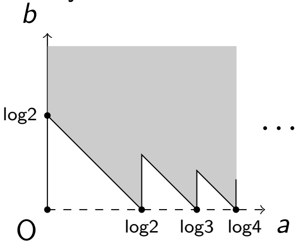


Figure 1: The region where $a + b \geq \log \lceil 2^a \rceil$.

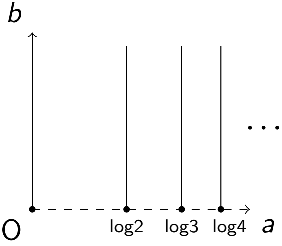


Figure 2: The region where $a = \log k$ for integer $k > 0, b \geq 0$.

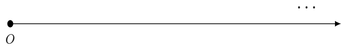
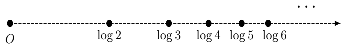
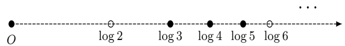
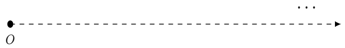
[1] František Matúš. "Piecewise linear conditional information inequality". In: *IEEE Transactions on Information Theory* 52.1 (2005), pp. 236–238.

[2] Qi Chen and Raymond W. Yeung. "Characterizing the entropy function region via extreme rays". In: *IEEE Information Theory Workshop*. Lausanne, Switzerland Sep. 2012. DOI: 10.1109/ITW.2012.6404674.

The extreme rays of Γ_4

- 41 extreme rays of Γ_4 can be classified into the following 11 types.
  - $U_{1,1}^{i}$, $i \in N_4$;
  - $U_{1,2}^{\alpha}$, $\alpha \subseteq N_4$, $|\alpha| = 2$;
  - $U_{1,3}^{\alpha}$, $\alpha \subseteq N_4$, $|\alpha| = 3$;
  - $U_{2,3}^{\alpha}$, $\alpha \subseteq N_4$, $|\alpha| = 3$;
  - $U_{1,4}$;
  - $\mathcal{W}_{2}^{\alpha}$, $\alpha \subseteq N_4$, $|\alpha| = 2$;
  - $U_{2,4}$;
  - $U_{3,4}$;
  - $\hat{U}_{2,5}^{i}$, $i \in N_4$;
  - $\hat{U}_{3,5}^{i}$, $i \in N_4$;
  - V_{8}^{α}, $\alpha \subseteq N_4$, $|\alpha| = 2$;

Entropy functions on the extreme rays of Γ_4

Extreme ray E	Entropy region $E^* = E \cap \Gamma_4^*$	Figures
$U_{1,1}^{1}, U_{1,2}^{12}, U_{1,3}^{123}, U_{1,4}$	$\{a\mathbf{r} : a \geq 0\}$	
$U_{2,3}^{123}, \mathcal{W}_{2}^{14}, U_{3,4}$	$\{a\mathbf{r} : a = \log k$ for some positive k $\in \mathbb{Z}\}$	
$U_{2,4}$	$\{a\mathbf{r} : a = \log k$ for some positive k $\in \mathbb{Z}, k \neq 2, 6\}$.	
$\hat{U}_{2,5}^{1}$	$\{a\mathbf{r} : a = \log k$ for some positive k $\in \mathbb{Z}\}$	
$\hat{U}_{3,5}^{1}$	$\{a\mathbf{r} : a = \log k$ for some positive k $\in \mathbb{Z}\}$	
V_{8}^{12}	$\{a\mathbf{r} : a = 0\}$	

Two-dimensional Face of Γ_4 Generating Algorithm

Input: The family $\mathcal{F}$ of all 28 facets and the family $\mathcal{E}$ of all 41 extreme rays of Γ_4.

Output: Upper triangle of a 41×41 $(0,1)$-matrix C,where $C(i,j) = 1$ if and component if the convex hull of the i-th extreme ray E_i and the j-th extreme ray E_j forms a 2-dimensional face of Γ_4.

for $1 \le i < j \le 41$ **do**
    $C(i,j) \leftarrow 1$
    for $k = 1$ to 28 **do**
        if the k-th facet F_k contains both E_i and E_j, **then**
            put F_k in $\mathcal{F}'$.
        end if
    end for
    for $E \in \mathcal{E} \setminus \{E_i, E_j\}$ **do**
        if E is contained in the face $\cap_{F \in \mathcal{F}'} F$ **then**
            $C(i,j) \leftarrow 0$
            break
        end if
    end for
end for

A catalogue of two-dimensional faces of Γ_4 (59 types)

	$U^i_{1,1}$	$U^\alpha_{1,2}$	$U^\alpha_{1,3}$	$U_{1,4}$	$U^\alpha_{2,3}$	$\mathcal{W}^\alpha_2$	$U_{2,4}$	$U_{3,4}$	$\hat{U}^i_{2,5}$	$\hat{U}^i_{3,5}$	V^α_8
$U^j_{1,1}$	$(U^1_{1,1}, U^2_{1,1})$ 6	$(U^{12}_{1,2}, U^1_{1,1})$ 12 $(U^{12}_{1,2}, U^3_{1,1})$ 12	$(U^{123}_{1,3}, U^1_{1,1})$ 12, $(U^{123}_{1,3}, U^4_{1,1})$ 4	$(U_{1,4}, U^1_{1,1})$ 4	$(U^{123}_{2,3}, U^1_{1,1})$ 12 $(U^{123}_{2,3}, U^4_{1,1})$ 4	$(\mathcal{W}^{14}_2, U^1_{1,1})$ 12 $(\mathcal{W}^{34}_2, U^1_{1,1})$ 12	$(U_{2,4}, U^1_{1,1})$ 4	$(U_{3,4}, U^1_{1,1})$ 4	$(\hat{U}^1_{2,5}, U^1_{1,1})$ 4 $(\hat{U}^1_{2,5}, U^2_{1,1})$ 12	$(\hat{U}^1_{3,5}, U^1_{1,1})$ 4 $(\hat{U}^1_{3,5}, U^2_{1,1})$ 12	$(V^{12}_8, U^1_{1,1})$ 12 $(V^{12}_8, U^3_{1,1})$ 12
$U^\beta_{1,2}$	\	$(U^{12}_{1,2}, U^{13}_{1,2})$ 12 $(U^{12}_{1,2}, U^{34}_{1,2})$ 3	$(U^{123}_{1,3}, U^{12}_{1,2})$ 12 $(U^{123}_{1,3}, U^{14}_{1,2})$ 12	$(U_{1,4}, U^{12}_{1,2})$ 6	$(U^{123}_{2,3}, U^{12}_{1,2})$ 12 $(U^{123}_{2,3}, U^{14}_{1,2})$ 12	$(\mathcal{W}^{14}_2, U^{14}_{1,2})$ 6 $(\mathcal{W}^{24}_2, U^{14}_{1,2})$ 24 $(\mathcal{W}^{34}_2, U^{12}_{1,2})$ 6	$(U_{2,4}, U^{12}_{1,2})$ 6	$(U_{3,4}, U^{12}_{1,2})$ 6	$(\hat{U}^1_{2,5}, U^{12}_{1,2})$ 12	$(\hat{U}^1_{3,5}, U^{12}_{1,2})$ 12	$(V^{12}_8, U^{13}_{1,2})$ 24
$U^\beta_{1,3}$	\	\	$(U^{123}_{1,3}, U^{124}_{1,3})$ 6	$(U_{1,4}, U^{123}_{1,3})$ 4	$(U^{123}_{2,3}, U^{124}_{1,3})$ 12	$(\mathcal{W}^{14}_2, U^{124}_{1,3})$ 12	$(U_{2,4}, U^{123}_{1,3})$ 4	$(U_{3,4}, U^{123}_{1,3})$ 4	$(\hat{U}^1_{2,5}, U^{123}_{1,3})$ 12	$(\hat{U}^1_{3,5}, U^{234}_{1,3})$ 4	$(V^{12}_8, U^{134}_{1,3})$ 12
$U_{1,4}$	\	\	\	\	$(U^{123}_{2,3}, U_{1,4})$ 4	0	0	$(U_{3,4}, U_{1,4})$ 1	0	0	$(V^{12}_8, U_{1,4})$ 6
$U^\beta_{2,3}$	\	\	\	\	$(U^{123}_{2,3}, U^{124}_{2,3})$ 6	$(\mathcal{W}^{12}_2, U^{134}_{2,3})$ 12	$(U_{2,4}, U^{123}_{2,3})$ 4	$(U_{3,4}, U^{123}_{2,3})$ 4	$(\hat{U}^1_{2,5}, U^{234}_{2,3})$ 4	$(\hat{U}^1_{3,5}, U^{123}_{2,3})$ 12	$(V^{12}_8, U^{123}_{2,3})$ 12
$\mathcal{W}^\beta_2$	\	\	\	\	\	$(\mathcal{W}^{12}_2, \mathcal{W}^{13}_2)$ 12	$(U_{2,4}, \mathcal{W}^{12}_2)$ 6	0	$(\hat{U}^1_{2,5}, \mathcal{W}^{12}_2)$ 12	$(\hat{U}^1_{3,5}, \mathcal{W}^{23}_2)$ 12	0
$U_{2,4}$	\	\	\	\	\	\	\	0	$(\hat{U}^1_{2,5}, U_{2,4})$ 4	$(\hat{U}^1_{3,5}, U_{2,4})$ 4	0
$U_{3,4}$	\	\	\	\	\	\	\	\	0	0	$(V^{12}_8, U_{3,4})$ 6
$\hat{U}^j_{2,5}$	\	\	\	\	\	\	\	\	0	0	0
$\hat{U}^j_{3,5}$	\	\	\	\	\	\	\	\	\	0	0
V^β_8	\	\	\	\	\	\	\	\	\	\	0

All-entropic and non-entropic faces

Theorem 1 (13 types)

For $F = (E_1, E_2)$, where distinct $E_i, i = 1, 2$ contains a rank-1 matroid, any $\mathbf{h} \in F$ is entropic.

[3] Randall Dougherty, Chris Freiling, and Kenneth Zeger. *Non-Shannon Information Inequalities in Four Random Variables*. 2011. arXiv: 1104.3602 [cs.IT]

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For $F = (E_1, E_2)$, where distinct $E_i, i = 1, 2$ contains a rank-1 matroid, any $\mathbf{h} \in F$ is entropic.

Theorem 2 (7 types,[3])

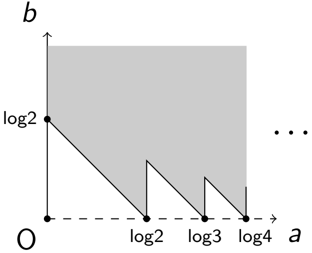
For $F = (V_8^{12}, E)$, any $\mathbf{h} = (a, b) \in F$ is non-entropic if a and b are both positive.

[3] Randall Dougherty, Chris Freiling, and Kenneth Zeger. *Non-Shannon Information Inequalities in Four Random Variables*. 2011. arXiv: 1104.3602 [cs.IT]

Extensions from 2-dim faces of Γ_3

Theorem 3 (4 types)

For $F = (U^{123}_{2,3}, U^{12}_{1,2}), (\mathcal{W}^{34}_2, U^{12}_{1,2}), (\mathcal{W}^{14}_2, U^{124}_{1,3})$, *or* $(\hat{U}^{1}_{2,5}, U^{123}_{1,3})$, $\mathbf{h} = (a, b) \in F$ *is entropic if and only if* $a + b \geq \log\lceil 2^a \rceil$.

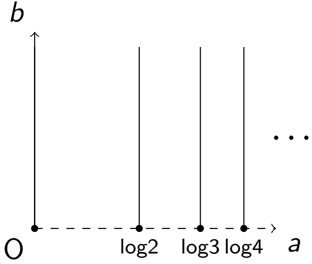


The entropy functions on the faces of these cases have the same shape as the two-dimensional face $(U_{2,3}, U^{12}_{1,2})$ of Γ_3.

Extension from 2-dimensional faces of Γ_3

Theorem 4 (13 types)

For $F = (U_{2,3}^{123}, U_{1,1}^{1}),(U_{2,3}^{123}, U_{1,1}^{4}), (U_{2,3}^{123}, U_{1,2}^{14}),(\mathcal{W}_2^{14}, U_{1,1}^{1}),(\mathcal{W}_2^{34}, U_{1,1}^{1}),(\mathcal{W}_2^{14}, U_{1,2}^{14}),$ $(\mathcal{W}_2^{24}, U_{1,2}^{14}),(\hat{U}_{2,5}^{1}, U_{1,1}^{1}),(\hat{U}_{2,5}^{1}, U_{1,1}^{2}),(\hat{U}_{2,5}^{1}, U_{1,2}^{12}),(\hat{U}_{3,5}^{1}, U_{1,1}^{1}), (\hat{U}_{3,5}^{1}, U_{1,1}^{2})$, *and* $(\hat{U}_{3,5}^{1}, U_{1,2}^{12})$ $\mathbf{h} = (a, b) \in F$ *is entropic if and only if* $a = \log k$ *for integer* $k > 0$.



The entropy functions of the faces on these cases have the same shape as the two-dimensional face $(U_{2,3}, U_{1,1}^{1})$ of Γ_3.

How about the faces containing $U_{2,4}$?

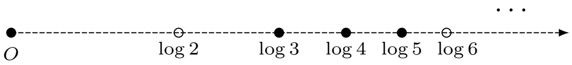


Figure 3: $E^*_{U_{2,4}} := \{a \cdot \mathbf{r}_{U_{2,4}} : a = \log k, k \neq 2, 6\ k \in \mathbb{Z}^+\}$

How about the faces containing $U_{2,4}$?

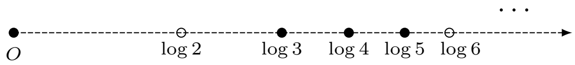


Figure 3: $E^*_{U_{2,4}} := \{a \cdot \mathbf{r}_{U_{2,4}} : a = \log k, k \neq 2, 6\ k \in \mathbb{Z}^+\}$

Characterizing random vector $(X_i, i \in N_4)$ satisfies

- $X_i \perp X_j$ for each $1 \leq i < j \leq 4$
- X_k is a function of X_i and X_j for any $1 \leq i < j \leq 4$ and $k \in \{1,2,3,4\} \setminus \{i,j\}$

Mutually orthogonal two latin squares

$$A := \begin{bmatrix} A & K & Q & J \\ Q & J & A & K \\ J & Q & K & A \\ K & A & J & Q \end{bmatrix}, \qquad B := \begin{bmatrix} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ \diamondsuit & \clubsuit & \spadesuit & \heartsuit \end{bmatrix}$$

- Two latin squres, each pair of symbols occurs exactly once.
- X_1, X_2, X_3 and X_4 are uniformly distributed on the rows, columns, symbols of the first square and symbols of the second square, respectively.

Mutually orthogonal two latin squares

$$A := \begin{bmatrix} A & K & Q & J \\ Q & J & A & K \\ J & Q & K & A \\ K & A & J & Q \end{bmatrix}, \qquad B := \begin{bmatrix} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ \diamondsuit & \clubsuit & \spadesuit & \heartsuit \end{bmatrix}$$

- Two latin squres, each pair of symbols occurs exactly once.
- X_1, X_2, X_3 and X_4 are uniformly distributed on the rows, columns, symbols of the first square and symbols of the second square, respectively.

For this case, $k \neq 2, 6$

- $k \neq 2$: trivial
- $k \neq 6$: Euler's 36 officer problem

Orthogonal array

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{matrix}$$

is an $\mathrm{OA}(2,4,3)$ corresponding to the MOLS.

Variable-strength orthogonal array(VOA)

Definition 1 ([4],[5])

Given a loopless matroid $M = (N_n, \mathbf{r})$ *with* $\mathbf{r}(N_n) \geq 2$, *a* $k^{\mathbf{r}(N_n)} \times n$ *array* $\mathbf{T}$

- *with columns index by* N_n,
- *entries from* N_k,

is called a variable-strength orthogonal array(VOA) *induced by* M *with level* k *if for any* $A \subseteq N_n$, $k^{\mathbf{r}(N_n)} \times |A|$ *subarry of* $\mathbf{T}$ *consisting of columns indexed by* A *satisfy the following condition:*

- *each row of this subarray occurs* $k^{\mathbf{r}(N_n)-\mathbf{r}(A)}$ *times.*

We call such $\mathbf{T}$ *a* $\mathrm{VOA}(M, k)$.

[4] Qi Chen, Minquan Cheng, and Baoming Bai. "Matroidal entropy functions: a quartet of theories of information, matroid, design and coding". In: *Entropy* 23.3 (2021), pp. 1–11

[5] Q. Chen, M. Cheng, and B. Bai. "Matroidal Entropy Functions: Constructions, Characterizations and Representations". In: *IEEE Transactions on Information Theory* (2024), pp. 1–1. DOI: 10.1109/TIT.2024.3355942

Variable-strength orthogonal array(VOA)

Theorem 5 ([4],[5])

A random vector $\mathbf{X} = (X_i : i \in N_n)$ *characterizes the matroidal entropy function* $\log k \cdot M$ *for a connected matroid* $M = (N_n, \mathbf{r})$ *with rank* $\mathbf{r}(N_n) \geq 2$ *if and only if the random variable* $\mathbf{X}$ *is uniformly distributed on the rows of a* $\mathrm{VOA}(M, k)$.

[4] Qi Chen, Minquan Cheng, and Baoming Bai. "Matroidal entropy functions: a quartet of theories of information, matroid, design and coding". In: *Entropy* 23.3 (2021), pp. 1–11

[5] Q. Chen, M. Cheng, and B. Bai. "Matroidal Entropy Functions: Constructions, Characterizations and Representations". In: *IEEE Transactions on Information Theory* (2024), pp. 1–1. DOI: 10.1109/TIT.2024.3355942

Theorem 6

For $F = (U_{2,4}, U_{2,3}^{123})$, $\mathbf{h} = (a, b) \in F$ *is entropic if and only if* $a + b = \log k$, $a = H(\alpha)$ *and* $(a, b) \neq (\log 2, 0), (\log 6, 0)$, *where integer* $k > 0$ *and* α *is a partition of* k.

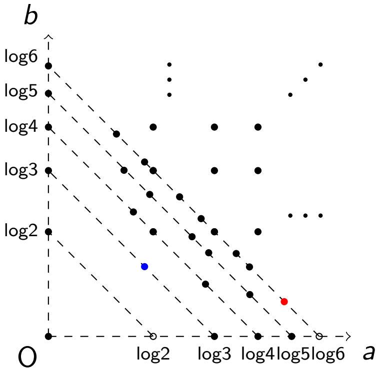


Figure 4: The face $(U_{2,4}, U_{2,3}^{123})$

$$
\begin{aligned}
H(X_{N_4}) &= H(X_{N_4-i}),\ \ i \in N_4 \\
H(X_{ij}) &= H(X_i) + H(X_j), i < j, i, j \in N_4, \\
H(X_{i\cup K}) + H(X_{j\cup K}) &= H(X_K) + H(X_{ij\cup K}), \\
|K| &= 2, K \subseteq \{1,2,3\}, \{i,j\} = N_4 \setminus K.
\end{aligned}
$$

which imply that

- $X_i, i = 1, 2, 3$ are uniformly distributed on N_k, and
- the distribution of X_4 can be any $\frac{\alpha}{k}$, where α is a number partition of k.

Semi-$\mathrm{VOA}(U_{2,4}, k)$ induced by a partition p of N_k

$\mathbf{h} = (a, b)$ with $a + b = \log 3$ and $a = H(\frac{1}{3}, \frac{2}{3})$

$$\mathrm{VOA}(U_{2,4}, 3)\ \mathbf{T} : \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \\ 2 & 1 & 3 & 3 \\ 2 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 3 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 \end{matrix} \qquad \mathbf{T}_p : \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 2 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 3 & 2 & 3 & 1 \\ 3 & 2 & 1 & 2 \end{matrix}$$

where $p = \{\{1\}, \{2, 3\}\}$. Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}_p$, then

- $a = H(X_4) = H(\frac{1}{3}, \frac{2}{3})$,
- $a + b = H(X_1) = H(X_2) = H(X_3) = \log 3$.

Almost $\mathrm{VOA}(U_{2,4}, k)$

Definition 2

For $k^2 \times 4$ array $\mathbf{T}$, it is called an almost $\mathrm{VOA}(U_{2,4}, k)$ if both $\mathbf{T}(1, 2, 3)$ and $\mathbf{T}(1, 2, 4)$ are $\mathrm{VOA}(U_{2,3}, k)$.

An almost-VOA$(U_{2,4}, 6)$[3]

$\mathbf{T}^{\mathrm{al}}$:

1	1	1	1	3	1	3	3	5	1	5	5
1	2	2	6	3	2	4	5	5	2	1	4
1	3	3	4	3	3	1	2	5	3	2	3
1	4	4	5	3	4	2	6	5	4	6	1
1	5	5	3	3	5	6	4	5	5	4	2
1	6	6	2	3	6	5	1	5	6	3	6
2	1	2	2	4	1	4	4	6	1	6	6
2	2	3	1	4	2	6	3	6	2	5	2
2	3	6	5	4	3	5	6	6	3	4	1
2	4	5	4	4	4	3	2	6	4	1	3
2	5	1	6	4	5	2	1	6	5	3	5
2	6	4	3	4	6	1	5	6	6	2	4

[3] Leonhard Euler. "Recherches sur un nouvelle espéce de quarrés magiques". In: *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen* (1782), pp. 85–239.

Semi-$\mathrm{VOA}(U_{2,4}, 6)$

-
  - $\mathbf{T}^{\mathrm{al}}(1,2,3)$ and $\mathbf{T}^{\mathrm{al}}(1,2,4)$ are both $\mathrm{VOA}(U_{2,3}, 6)$.
  - $\mathbf{T}$ is not a $\mathrm{VOA}(U_{2,4}, 6)$ since there are 34 different pairs in the rows of $\mathbf{T}^{\mathrm{al}}(\{3,4\})$, where $(2,6)$ and $(4,5)$ each occurs twice.

Semi-$\mathrm{VOA}(U_{2,4}, 6)$

- - $\mathbf{T}^{\mathrm{al}}(1,2,3)$ and $\mathbf{T}^{\mathrm{al}}(1,2,4)$ are both $\mathrm{VOA}(U_{2,3}, 6)$.
  - $\mathbf{T}$ is not a $\mathrm{VOA}(U_{2,4}, 6)$ since there are 34 different pairs in the rows of $\mathbf{T}^{\mathrm{al}}(\{3,4\})$, where $(2,6)$ and $(4,5)$ each occurs twice.
- - Consider a partition $p = \{\{1\},\{2\},\{3\},\{4\},\{5,6\}\}$ of N_6.
  - Let $\mathbf{T}_p$ be a 36×4 array such that $\mathbf{T}_p(N_3) = \mathbf{T}^{\mathrm{al}}(N_3)$ and each entry $\mathbf{T}_p(4)$ follows the mapping from those $\mathbf{T}^{\mathrm{al}}(4)$

$$\begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto 2 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4 \\ 5,6 &\mapsto 5 \end{aligned}$$

Semi-$\mathrm{VOA}(U_{2,4}, 6)$

- - $\mathbf{T}^{\mathrm{al}}(1,2,3)$ and $\mathbf{T}^{\mathrm{al}}(1,2,4)$ are both $\mathrm{VOA}(U_{2,3}, 6)$.
  - $\mathbf{T}$ is not a $\mathrm{VOA}(U_{2,4}, 6)$ since there are 34 different pairs in the rows of $\mathbf{T}^{\mathrm{al}}(\{3,4\})$, where $(2,6)$ and $(4,5)$ each occurs twice.
- - Consider a partition $p = \{\{1\},\{2\},\{3\},\{4\},\{5,6\}\}$ of N_6.
  - Let $\mathbf{T}_p$ be a 36×4 array such that $\mathbf{T}_p(N_3) = \mathbf{T}^{\mathrm{al}}(N_3)$ and each entry $\mathbf{T}_p(4)$ follows the mapping from those $\mathbf{T}^{\mathrm{al}}(4)$

$$\begin{aligned} 1 &\mapsto 1 \\ 2 &\mapsto 2 \\ 3 &\mapsto 3 \\ 4 &\mapsto 4 \\ 5,6 &\mapsto 5 \end{aligned}$$

- for a partiton p' coarser than p, we can obtain a $\mathbf{T}_{p'}$ similarly

$\mathbf{T}_p$:

1	1	1	1	3	1	3	3	5	1	5	5
1	2	2	5	3	2	4	5	5	2	1	4
1	3	3	4	3	3	1	2	5	3	2	3
1	4	4	5	3	4	2	5	5	4	6	1
1	5	5	3	3	5	6	4	5	5	4	2
1	6	6	2	3	6	5	1	5	6	3	5
2	1	2	2	4	1	4	4	6	1	6	5
2	2	3	1	4	2	6	3	6	2	5	2
2	3	6	5	4	3	5	5	6	3	4	1
2	4	5	4	4	4	3	2	6	4	1	3
2	5	1	5	4	5	2	1	6	5	3	5
2	6	4	3	4	6	1	5	6	6	2	4

$\mathbf{T}_p$:

1	1	1	1	3	1	3	3	5	1	5	5
1	2	2	5	3	2	4	5	5	2	1	4
1	3	3	4	3	3	1	2	5	3	2	3
1	4	4	5	3	4	2	5	5	4	6	1
1	5	5	3	3	5	6	4	5	5	4	2
1	6	6	2	3	6	5	1	5	6	3	5
2	1	2	2	4	1	4	4	6	1	6	5
2	2	3	1	4	2	6	3	6	2	5	2
2	3	6	5	4	3	5	5	6	3	4	1
2	4	5	4	4	4	3	2	6	4	1	3
2	5	1	5	4	5	2	1	6	5	3	5
2	6	4	3	4	6	1	5	6	6	2	4

- Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}_p$ and the entropy function of $(X_i, i \in N_4)$ corresponds to the “red” polymatrioid in Fig.4.

$\mathbf{T}_p$:

1	1	1	1	3	1	3	3	5	1	5	5
1	2	2	5	3	2	4	5	5	2	1	4
1	3	3	4	3	3	1	2	5	3	2	3
1	4	4	5	3	4	2	5	5	4	6	1
1	5	5	3	3	5	6	4	5	5	4	2
1	6	6	2	3	6	5	1	5	6	3	5
2	1	2	2	4	1	4	4	6	1	6	5
2	2	3	1	4	2	6	3	6	2	5	2
2	3	6	5	4	3	5	5	6	3	4	1
2	4	5	4	4	4	3	2	6	4	1	3
2	5	1	5	4	5	2	1	6	5	3	5
2	6	4	3	4	6	1	5	6	6	2	4

- Let $(X_i, i \in N_4)$ be uniformly distributed on the rows of $\mathbf{T}_p$ and the entropy function of $(X_i, i \in N_4)$ corresponds to the "red" polymatrioid in Fig.4.
- Semi-VOA will shed light on open problems in combinatorial design theory.

Theorem 7

For $F = (U_{2,4}, \mathcal{W}_2^{34})$, $\mathbf{h} = (a, b) \in F$ *is entropic if and only if* $a + b = \log k$ *for integer* $k > 0$, *and there exists an almost* $\mathrm{VOA}(U_{2,4}, k)$ $\mathbf{T}$, *and*

$$a = H(\boldsymbol{\alpha}) - \log k,$$

where $\boldsymbol{\alpha} = (\alpha_{x_3,x_4} > 0 : x_3, x_4 \in N_k)$ *and* α_{x_3,x_4} *denotes the times of the row* (x_3, x_4) *that occurs in* $\mathbf{T}(3,4)$.

Entropy functions on $(U_{2,4}, \mathcal{W}_2^{34})$

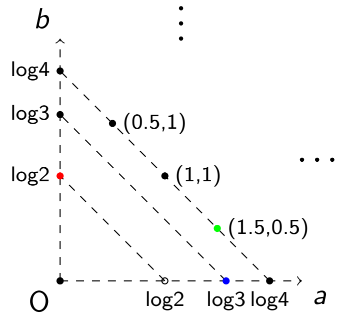


Figure 5: The face $(U_{2,4}, \mathcal{W}_2^{34})$

$\mathbf{T}_1$:

1	1	1	1
1	2	2	2
2	1	2	2
2	2	1	1

$\mathbf{T}_2$:

1	1	1	1
1	2	2	2
1	3	3	3
2	1	3	2
2	2	1	3
2	3	2	1
3	1	2	3
3	2	3	1
3	3	1	2

$\mathbf{T}_3$:

1	1	1	1
1	2	2	2
1	3	3	3
1	4	4	4
2	1	2	2
2	2	3	3
2	3	1	1
2	4	4	4
3	1	4	3
3	2	1	4
3	3	2	1
3	4	3	2
4	1	3	4
4	2	4	1
4	3	1	2
4	4	2	3

Uniform decomposition of a $\mathrm{VOA}(U_{2,3}, k)$

Definition 3

Given $A, B \subseteq N_k$ *and a* $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$, *a* $|A||B| \times 3$ *subarray* $\mathbf{T}'$ *of* $\mathbf{T}$ *is called induced by* A *and* B *if rows in* $\mathbf{T}'(1, 2)$ *are exactly those pairs in* $A \times B$.

Uniform decomposition of a $\mathrm{VOA}(U_{2,3}, k)$

Definition 3

Given $A, B \subseteq N_k$ and a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$, a $|A||B| \times 3$ subarray $\mathbf{T}'$ of $\mathbf{T}$ is called induced by A and B if rows in $\mathbf{T}'(1, 2)$ are exactly those pairs in $A \times B$.

Definition 4

Given $A, B \subseteq N_k$ with $|A||B| = k$ and a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$,

- *a subarray $\mathbf{T}'$ of $\mathbf{T}$ induced by A and B is called a unit subarray of $\mathbf{T}$ if each $e \in N_k$ occurs exactly once in $\mathbf{T}'(3)$.*
- *$\{\mathbf{T}_i, i \in N_k\}$ is called an uniform decomposition of a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ if*
  - *each $\mathbf{T}_i$ is a unit subarray of $\mathbf{T}$ and*
  - *$\biguplus_{i \in N_k} A_i \times B_i = N_k^2$.*

An example of uniform decomposition

$$\mathbf{T}: \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 4 & 4 \\ 3 & 1 & 3 \\ 3 & 2 & 1 \\ 3 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 1 & 4 \\ 4 & 2 & 2 \\ 4 & 3 & 3 \\ 4 & 4 & 1 \end{matrix}$$

$$\mathbf{T}_1: \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \end{matrix} \qquad \mathbf{T}_2: \begin{matrix} 2 & 1 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 4 & 4 \end{matrix} \qquad \mathbf{T}_3: \begin{matrix} 3 & 1 & 3 \\ 3 & 2 & 1 \\ 4 & 1 & 4 \\ 4 & 2 & 2 \end{matrix} \qquad \mathbf{T}_4: \begin{matrix} 3 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 3 & 3 \\ 4 & 4 & 1 \end{matrix}$$

$A_1 = \{1\}$, $B_1 = N_4$

$A_2 = \{2\}$, $B_2 = N_4$

$A_3 = \{3, 4\}$, $B_3 = N_2$

$A_4 = \{3, 4\}$, $B_4 = \{3, 4\}$

Theorem 8

For $F = (\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$, $\mathbf{h} = (a, b) \in F$ *is entropic if and only if there exists a uniform decomposition* $\{\mathbf{T}_1, \ldots, \mathbf{T}_k\}$ *of a* $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ *such that*

$$a = \log k - \frac{1}{k}\sum_{i=1}^{k} \log |B_i|, \quad b = \log k - \frac{1}{k}\sum_{i=1}^{k} \log |A_i|,$$

where the subarray $\mathbf{T}_i$ *of* $\mathbf{T}$ *are induced by* A_i *and* B_i *for* $i \in N_k$.

Entropy functions on $(\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$

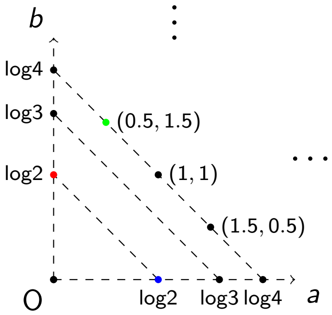


Figure 6: The face $(\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$

$$\mathbf{T}: \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{matrix}$$ can be decomposed into

- $\mathbf{T}_1 : \begin{matrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{matrix}$, $\mathbf{T}_2 : \begin{matrix} 1 & 2 & 2 \\ 2 & 2 & 1 \end{matrix}$ or
  $A_1 = \{1, 2\}$ $A_2 = \{1, 2\}$
  $B_1 = \{1\}$ $B_2 = \{2\}$
- $\mathbf{T}_1 : \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 2 \end{matrix}$, $\mathbf{T}_2 : \begin{matrix} 2 & 1 & 2 \\ 2 & 2 & 1 \end{matrix}$.
  $A_1 = \{1\}$ $A_2 = \{2\}$
  $B_1 = \{1, 2\}$ $B_2 = \{1, 2\}$

Definition 5

Given $A, B \subseteq N_k$ *with* $|A| = |B| = k_1 \leq k$ *and a* $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$,

- *a subarray* $\mathbf{T}'$ *of* $\mathbf{T}$ *induced by* A *and* B *is called a suborder* VOA *of* $\mathbf{T}$ *if* $\mathbf{T}'$ *is a* $\mathrm{VOA}(U_{2,3}, k_1)$.
- $\{\mathbf{T}_i, i \in N_t\}$ *is called a suborder decomposition of a* $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ *if*
  - *each* $\mathbf{T}_i$ *is a suborder* VOA *of* $\mathbf{T}$ *and*
  - $\biguplus_{i \in N_t} A_i \times B_i = N_k^2$.

An example of suborder decomposition

$$\mathbf{T}: \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 4 & 4 \\ 3 & 1 & 3 \\ 3 & 2 & 1 \\ 3 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 1 & 4 \\ 4 & 2 & 2 \\ 4 & 3 & 3 \\ 4 & 4 & 1 \end{matrix}$$

$$\mathbf{T}_1: \begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \quad \mathbf{T}_2: \begin{matrix} 1 & 2 & 4 \\ 1 & 4 & 3 \\ 2 & 2 & 3 \\ 2 & 4 & 4 \end{matrix} \quad \mathbf{T}_3: \begin{matrix} 3 & 1 & 3 \\ 3 & 3 & 4 \\ 4 & 1 & 4 \\ 4 & 3 & 3 \end{matrix}$$

$$\mathbf{T}_4: 3 \quad 2 \quad 1 \qquad \mathbf{T}_5: 3 \quad 4 \quad 2 \qquad \mathbf{T}_6: 4 \quad 2 \quad 2$$

$$\mathbf{T}_7: 4 \quad 4 \quad 1$$

Theorem 9

For $F = (\hat{U}^{1}_{2,5}, U^{234}_{2,3})$, $\mathbf{h} = (a, b) \in F$ is entropic if and only if $a + b = \log k$ for integer $k > 0$, and there exists a suborder decomposition $\{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_t\}$ of a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ such that

$$a = \frac{1}{2} H(\frac{|A_i|^2}{k^2} : i \in N_t),$$

where the subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by A_i and B_i for $i \in N_t$.

Entropy functions on $(\hat{U}^1_{2,5}, U^{234}_{2,3})$

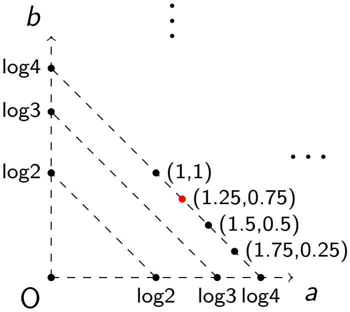
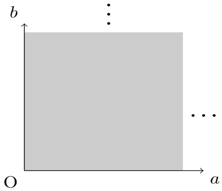
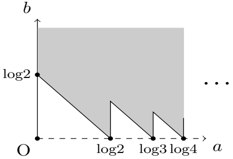
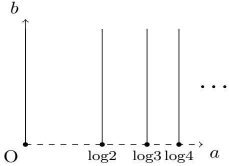
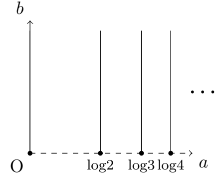
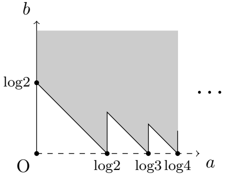
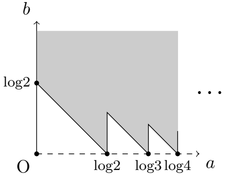


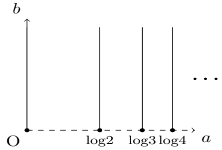
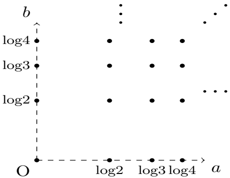
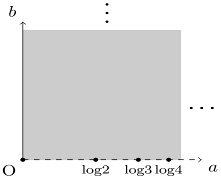
Figure 7: The face $(\hat{U}^1_{2,5}, U^{234}_{2,3})$

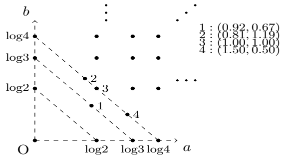
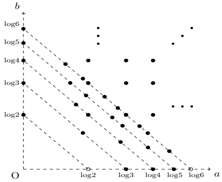
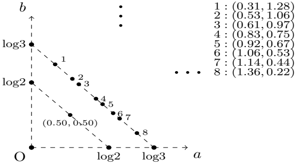
The "red" polymatroid corresponds to the suborder decomposition in the above example.

$$a = \frac{1}{2}H(\frac{4}{16}, \frac{4}{16}, \frac{4}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$$
$$= \frac{1}{2}[H(\frac{4}{16}, \frac{4}{16}, \frac{4}{16}, \frac{4}{16}) + \frac{1}{4}\log 4]$$
$$= \frac{5}{8}\log 4 = 1.25$$

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(U^1_{1,1}, U^2_{1,1})$, $(U^{12}_{1,2}, U^1_{1,1})$, $(U^{12}_{1,2}, U^3_{1,1})$, $(U^{12}_{1,2}, U^{13}_{1,2})$, $(U^{12}_{1,2}, U^{34}_{1,2})$, $(U^{123}_{1,3}, U^1_{1,1})$, $(U^{123}_{1,3}, U^4_{1,1})$, $(U^{123}_{1,3}, U^{12}_{1,2})$, $(U^{123}_{1,3}, U^{14}_{1,2})$, $(U^{123}_{1,3}, U^{124}_{1,3})$, $(U_{1,4}, U^1_{1,1})$, $(U_{1,4}, U^{12}_{1,2})$, $(U_{1,4}, U^{123}_{1,3})$.	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a \geq 0, b \geq 0\}$	
$(U^{123}_{2,3}, U^{12}_{1,2})$, $(\mathcal{W}^{34}_2, U^{12}_{1,2})$, $(\mathcal{W}^{14}_2, U^{124}_{1,3})$.	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b \geq \log k$ and $\log(k-1) \leq a \leq \log k$ for positive integer $k\}$	
$(U^{123}_{2,3}, U^1_{1,1})$, $(U^{123}_{2,3}, U^4_{1,1})$, $(U^{123}_{2,3}, U^{14}_{1,2})$, $(\mathcal{W}^{14}_2, U^1_{1,1})$, $(\mathcal{W}^{34}_2, U^1_{1,1})$, $(\mathcal{W}^{14}_2, U^{14}_{1,2})$, $(\mathcal{W}^{24}_2, U^{14}_{1,2})$.	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer k, $b \geq 0\}$	

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(\hat{U}^1_{2,5}, U^1_{1,1})$, $(\hat{U}^1_{2,5}, U^2_{1,1})$, $(\hat{U}^1_{2,5}, U^{12}_{1,2})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer k, $b \geq 0\}$	
$(\hat{U}^1_{2,5}, U^{123}_{1,3})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b \geq \log k$ and $\log(k-1) \leq a \leq \log k$ for positive integer $k\}$	
$(\hat{U}^1_{3,5}, U^1_{1,1})$, $(\hat{U}^1_{3,5}, U^2_{1,1})$, $(\hat{U}^1_{3,5}, U^{12}_{1,2})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b \geq \log k$ and $\log(k-1) \leq a \leq \log k$ for positive integer $k\}$	

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(U_{2,3}^{123}, U_{1,3}^{124})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer k, $b \geq 0\}$	
$(U_{2,3}^{123}, U_{2,3}^{124})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k_1, b = \log k_2$ for some positive integer k_1, $k_2\}$	
$(U_{2,3}^{123}, U_{1,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a \geq 0, b > 0$ or $(a, b) = (\log k, 0)$ for positive integer $k\}$	

<table>
<tr><th>Two-dimensional faces F</th><th>Entropy region $F^* = F \cap \Gamma_4^*$</th><th>Figures</th></tr>
<tr><td>$(\mathcal{W}_2^{12}, U_{2,3}^{134})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k, a = H(\boldsymbol{\alpha})$, where integer $k > 0$ and $\boldsymbol{\alpha}$ is a partition of $k\}$</td><td>
</td></tr>
<tr><td>$(U_{2,4}, U_{2,3}^{123})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k, a = H(\boldsymbol{\alpha})$ and $(a, b) \neq (\log 2, 0), (\log 6, 0)$, where integer $k > 0$ and $\boldsymbol{\alpha}$ is a partition of $k\}$</td><td>
</td></tr>
<tr><td>$(\hat{U}_{2,5}^{1}, \mathcal{W}_2^{12})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for some positive k and $a = \frac{1}{k}\sum_{i=1}^{k} H(\boldsymbol{\alpha_i})$, where $\boldsymbol{\alpha_i} \in \mathcal{P}(k), i = 1, 2, ..., k\}$</td><td>
</td></tr>
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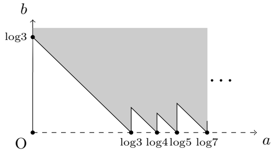
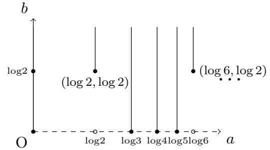
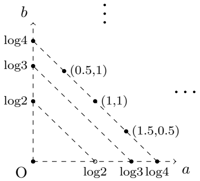
<table>
<tr><th>Two-dimensional faces F</th><th>Entropy region $F^* = F \cap \Gamma_4^*$</th><th>Figures</th></tr>
<tr><td>$(U_{2,4}, U_{1,3}^{123})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b \geq \log k$ and $\log(k-1) < a \leq \log k$ for positive integer $k \neq 2, 6$; or $a + b \geq \log(k+1)$ and $\log(k-1) < a \leq \log k$ for $k = 2, 6\}$</td><td>
</td></tr>
<tr><td>$(U_{2,4}, U_{1,2}^{12})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for positive integer $k \neq 2, 6$; $a = \log 2,\ b \geq \log 2$; or $a = \log 6,\ b \geq \log 2\}$</td><td>
</td></tr>
<tr><td>$(U_{2,4}, \mathcal{W}_2^{12})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for integer $k > 0$, and there exists a $k^2 \times 4$ array $\mathbf{T}$ such that $\mathbf{T}(1,3,4)$ and $\mathbf{T}(2,3,4)$ are $\mathrm{VOA}(U_{2,3}, k)$, and $a = H(\boldsymbol{\alpha}) - \log k$, where $\boldsymbol{\alpha} = (\alpha_{x_1,x_2} > 0 : x_1, x_2 \in N_k)$ and α_{x_1,x_2} denotes the times of the row (x_1, x_2) that occurs in $\mathbf{T}(1,2)\}$</td><td>
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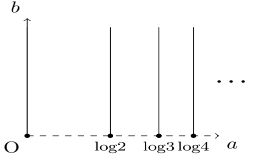
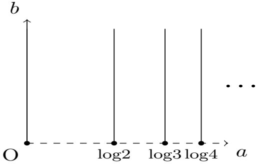
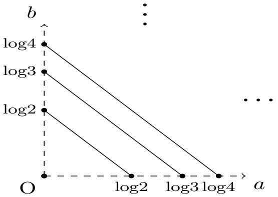
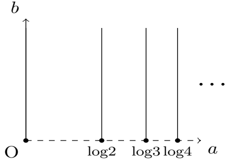
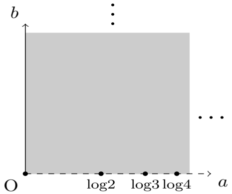
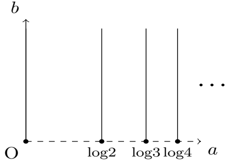
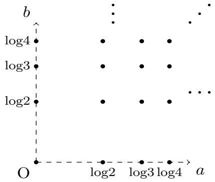
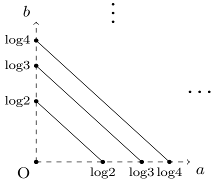
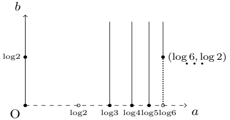
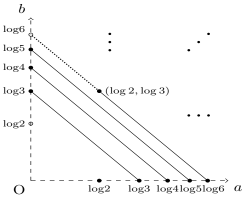
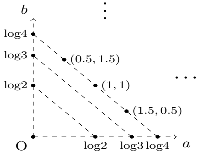
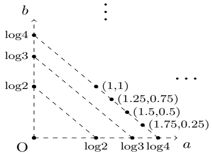
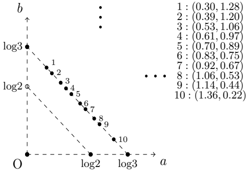
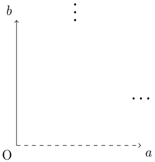
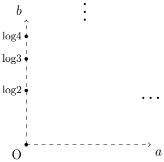
Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(U_{3,4}, U_{1,1}^{1})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer k, $b \geq 0\}$	
$(U_{3,4}, U_{1,2}^{12})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer k, $b \geq 0\}$	
$(U_{3,4}, U_{2,3}^{123})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for some positive integer $k\}$	

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(U_{3,4}, U_{1,3}^{123})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer $k,\ b \geq 0\}$	
$(U_{3,4}, U_{1,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a \geq 0, b > 0$ or $(a, b) = (\log k, 0)$ for positive integer $k\}$	
$(\hat{U}_{3,5}^{1}, U_{1,3}^{234})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for some positive integer $k,\ b \geq 0\}$	

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(\hat{U}^1_{3,5}, U^{123}_{2,3})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k_1, b = \log k_2$ for some positive integer $k_1,\ k_2\}$	
$(\hat{U}^1_{3,5}, \mathcal{W}^{23}_2)$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for some integer $k > 0\}$	
$(U_{2,4}, U^1_{1,1})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = \log k$ for integer $k \neq 2, 6$ or $a = \log 6, b \geq \log 2\} \subseteq F^*$ and $\{a\mathbf{r}_1 + b\mathbf{r}_2 : a \neq \log k$ for some integer $k > 0$ or $a = \log 2\} \cap F^* = \emptyset$	

<table>
<tr><th>Two-dimensional faces F</th><th>Entropy region $F^* = F \cap \Gamma_4^*$</th><th>Figures</th></tr>
<tr><td>$(\hat{U}^1_{3,5}, U_{2,4})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for integer $k \neq 2, 6$;
$(a, b) = (\log 2, 0)$; or
$a + b = \log 6, a \geq \log 2\} \subseteq F^*$ and
$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b \neq \log k$ for some integer $k > 0$;
$a + b = \log 2,\ a < \log 2$; or
$(a, b) = (0, \log 6)\} \cap F^* = \emptyset$.</td><td>
</td></tr>
<tr><td>$(\mathcal{W}_2^{12}, \mathcal{W}_2^{13})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2$: there exists an entry-subarray decomposition
$\{\mathbf{T}_1, \ldots, \mathbf{T}_k\}$ of a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ such that
$a = \log k - \frac{1}{k}\sum_{i=1}^{k} \log \lvert B_i \rvert$,
$b = \log k - \frac{1}{k}\sum_{i=1}^{k} \log \lvert A_i \rvert$,
where the subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by
A_i and B_i for $i \in N_k\}$</td><td>
</td></tr>
<tr><td>$(\hat{U}^1_{2,5}, U^{234}_{2,3})$</td><td>$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ for some positive k and
there exists a VOA decomposition $\{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_t\}$
of a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}$ such that
$a = \frac{1}{2}H(\frac{\lvert A_i \rvert^2}{k^2} : i \in N_t)$,
where subarray $\mathbf{T}_i$ of $\mathbf{T}$ are induced by
A_i and B_i for $i \in N_t\}$</td><td>
</td></tr>
</table>

Two-dimensional faces F	Entropy region $F^* = F \cap \Gamma_4^*$	Figures
$(\hat{U}_{2,5}^1, U_{2,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a + b = \log k$ and there exists a $\mathrm{VOA}(U_{2,3}, k)$ $\mathbf{T}'$ and its loose orthogonal array $\mathbf{T}_1$ such that $a = H(\frac{\alpha_1}{k^2}, \frac{\alpha_2}{k^2}, \ldots, \frac{\alpha_t}{k^2}) - \log k$, where α_i denotes the times of the row x_1 that occurs in $\mathbf{T}_1\}$	
$(V_8^{12}, U_{1,1}^1)$, $(V_8^{12}, U_{1,1}^3)$, $(V_8^{12}, U_{1,2}^{12})$ $(V_8^{12}, U_{1,3}^{134})$, $(V_8^{12}, U_{1,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = 0, b \geq 0\}$	
$(V_8^{12}, U_{2,3}^{123})$, $(V_8^{12}, U_{3,4})$	$\{a\mathbf{r}_1 + b\mathbf{r}_2 : a = 0, b = \log k$ for some integer $k > 0\}$	

Shaocheng Liu

Minquan Cheng

[1] S. Liu and Q. Chen. "Entropy Functions on Two-Dimensional Faces of Polymatroidal Region of Degree Four". In: *2023 IEEE International Symposium on Information Theory (ISIT)*. 2023

[2] S. Liu and Q. Chen. "Entropy Functions on Two-Dimensional Faces of Polymatroidal Region of Degree Four: Part I: Problem Formulation and Graph-Coloring Approach". In preparation

[3] S. Liu, Q. Chen and M. Cheng. "Entropy Functions on Two-Dimensional Faces of Polymatroidal Region of Degree Four: Part II: Information Theoretic Constraints Breed New Combinatorial Structures". In preparation

Thank You!