Entropy and matroids

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collect basic properties of the ranks of all A_I submatrices

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crossroad of combinatorics, algebra and finite geometry

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for $K \subseteq N$ all $I \subseteq K$ maximal in $\mathcal{I}$ have the same cardinality.

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many theorems on one-to-one correspondences

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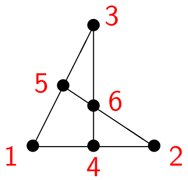

$r(I)$ ... affine dimension of I
B base ... three points not collinear
C circuit ... $\{1, 2, 4\}$, $\{1, 4, 5, 6\}$, etc.

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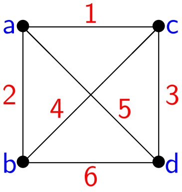

B base ... edges of a spanning tree
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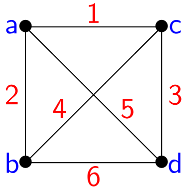

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matroids ... vertexless graphs

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... an array of joint outcomes
with a column of probabilities
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look at the entropy $H(\xi_I)$

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... three binary variables

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$H(\xi_\emptyset)$	$H(\xi_1)$	$H(\xi_2)$	$H(\xi_3)$	$H(\xi_1,\xi_2)$	$H(\xi_1,\xi_3)$	$H(\xi_2,\xi_3)$	$H(\xi_1,\xi_2,\xi_3)$
0	$\ln 2$	$\ln 2$	$\ln 2$	$\ln 4$	$\ln 4$	$\ln 4$	$\ln 4$

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… three binary variables

$H(\xi_\emptyset)$	$H(\xi_1)$	$H(\xi_2)$	$H(\xi_3)$	$H(\xi_1,\xi_2)$	$H(\xi_1,\xi_3)$	$H(\xi_2,\xi_3)$	$H(\xi_1,\xi_2,\xi_3)$
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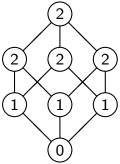

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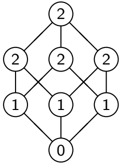
matroidal rank function 

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Theorem

If $\xi_1, \ldots, \xi_n$ are distributed uniformly on the linear code generated by a matrix A over $\mathbb{F}$ then $H(\xi_I) = \mathrm{rank}(A_I) \ln |\mathbb{F}|$, $I \subseteq N$.

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connections to entropy: Fujishige 1978, Pippenger 1986

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an analogous theorem identifying matroids in the network coding?

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N. Ay 2002

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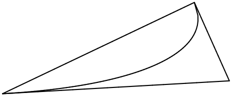

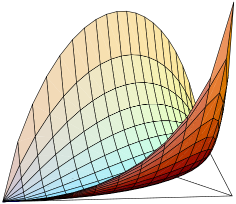

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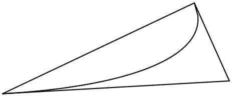

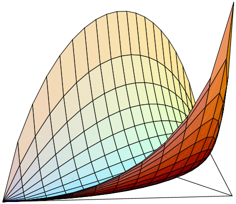

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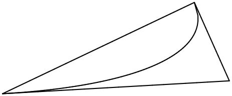

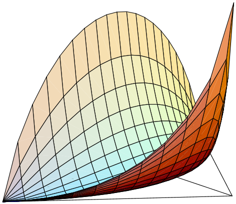

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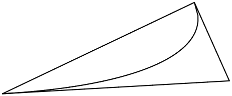

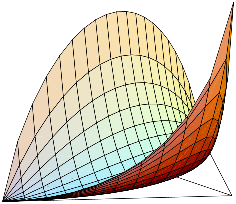

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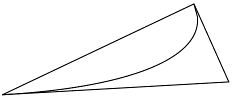

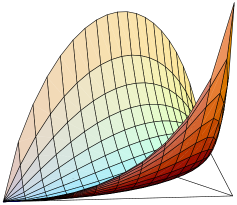

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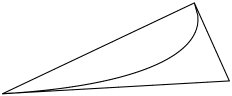

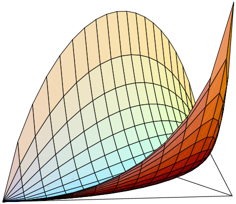

when the model consists of the factorizable distributions over a hypergraph (hierarchical log-linear models, in particular graphical Markov ones) then sometimes maximizers correspond to the ideal sss's (FM 2009)

If a matroid (N, r) is linear over a field $\mathbb{F}$
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Given a matroid (N, r) and $t \geqslant 0$, a random vector $(\xi_i)_{i \in N}$ represents $(N, r \cdot t)$ iff $t = \ln d$ for some integer $d \geqslant 1$ and ξ_I takes $d^{r(I)}$ values with the same probability, $I \subseteq N$.

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… secret sharing matroids, almost affine codes

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11	23	32
33	12	21
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array of nine four-tuples $\{(i, j, k, l) \colon i, j \in \{1, 2, 3\}\}$

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A matroid (N, r) is p-representable of the degree $2/3$ iff it is linear over $\mathrm{GF}(2)/\mathrm{GF}(3)$. (Beimel, FM, independently)

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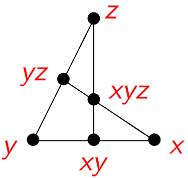

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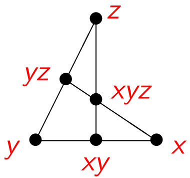

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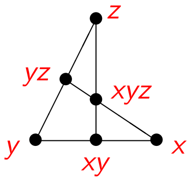

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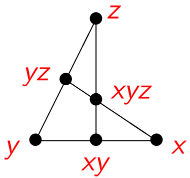

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… an equivalent definition of the group via entropy

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? can a p-representable matroid violate Ingleton inequality ?