Entropic Vectors: Polyhedral Computation & Information Geometry

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Outline

1. Entropic Vectors and Polyhedral Computation
   (a) One Procedure to Rule Four Problems
      i. Non-Shannon outer bounds for $\bar{\Gamma}^*_N$ (Yeung, Zhang, DFZ, Matúš, ...)
      ii. Vector Matroidal Inner bound for $\bar{\Gamma}^*_N$ (Hassibi)
      iii. Linear Code/Subspace Rank Outer Bound (Hammer, Kinser, DFZ)
      iv. Network Coding/Distributed Storage Regions (Yan, Yeung, Chan, Grant)
   (b) Many Paths Lead to the Same Truth
   (c) The Path Less Laden: Complexity Experiments & Moving Forward
2. Structure of $\bar{\Gamma}^*_4$ & the Case for Non-Polyhedral Tools
3. Characterizing Extremal Entropic Vectors with Information Geometry
   (a) Introduction to Information Geometry
   (b) Decoupling Constraints & Improving Ingleton w/ Binary Inner Bound
   (c) Casting Entropic Vectors as Information Projections
   (d) Information Geometric Properties of Distributions on Shannon Facets
   (e) $\bar{\Gamma}^*_4$ & 4 atom distribution. Information Geometry of Ingleton Violation

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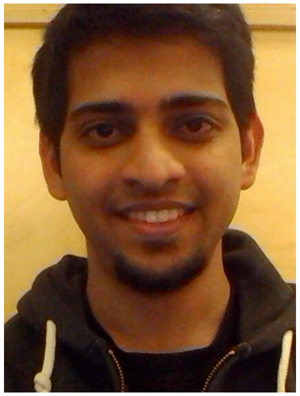
Entropic Vectors and Polyhedral Computation
(Development Team)



Jayant Apte

efficient parallel infinite precision polyhedral computation

Congduan Li

rate regions & rate delay tradeoffs

Daniel Venutolo

computing non-Shannon inequalities

Prof. Steven Weber

Drexel University

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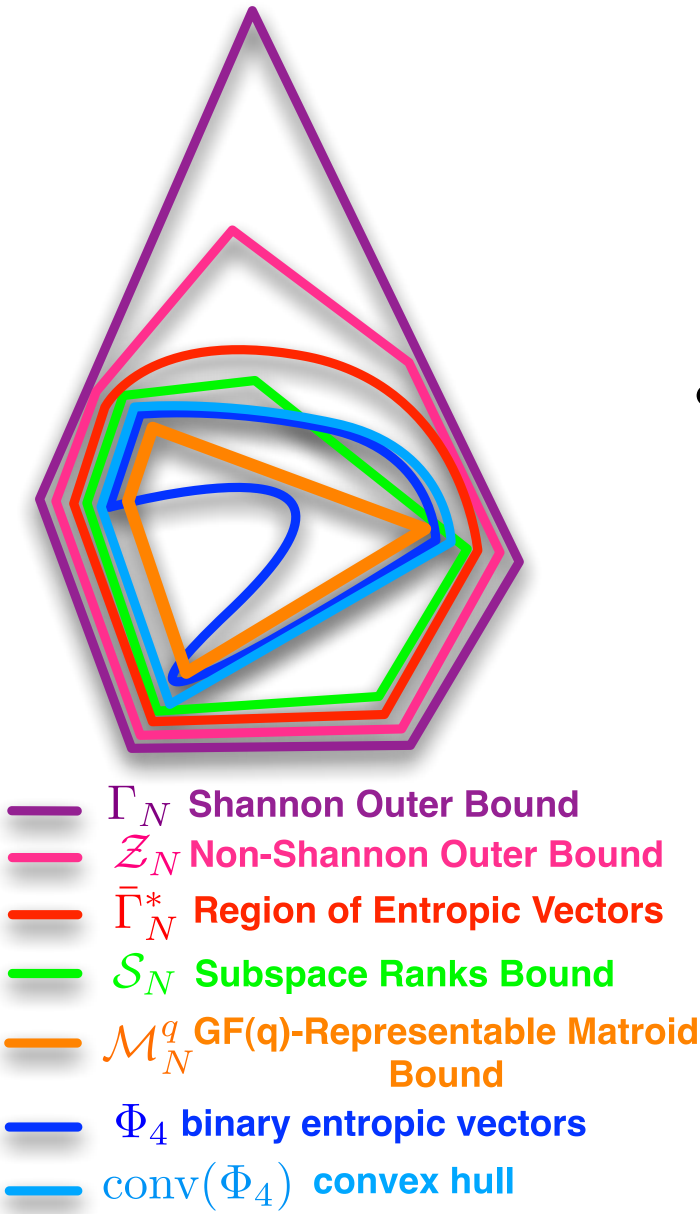
Bounding the Region of Entropic Vectors $\bar{\Gamma}_N^*$ from the Outside

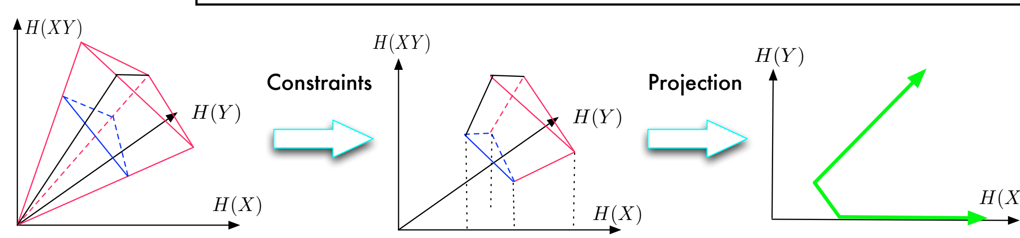


- **Shannon Outer Bound**: Γ_N. entropy is submodular:

$$I(\boldsymbol{X}_{\mathcal{A}}; \boldsymbol{X}_{\mathcal{B}} | \boldsymbol{X}_{\mathcal{C}}) \geq 0 \quad \forall \mathcal{A}, \mathcal{B}, \mathcal{C}$$

  $\Gamma_2 = \bar{\Gamma}_2^*, \Gamma_3 = \bar{\Gamma}_3^*$.
  $\Gamma_N \neq \bar{\Gamma}_N^*, N \geq 4$ $\bar{\Gamma}_N^*$ non-polyhedral convex cone

- **Non-Shannon Outer Bounds**:[1, 2, 3, 4, 5, 6, 7]
  Yeung & Zhang, Dougherty & Freiling & Zeger, Matus
  Start with 4 unconstr. r.v.s
  add rv. obeying distr. match & Markov. cond.
  Intersect Γ_N for $N \geq 5$ w/ Markov & distr. match
  Project back to orig. 4 unconstr. vars.
  obtain new information inequalities this way!
  overall: Shannon $\rightarrow$ linear eq./ineq. $\bigcap$ $\rightarrow$ project



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Bounding $\bar{\Gamma}^*_N$ from the Inside, 1: Representable Matroids

Matroids: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$, $r(\mathcal{A}) \leq |\mathcal{A}|$

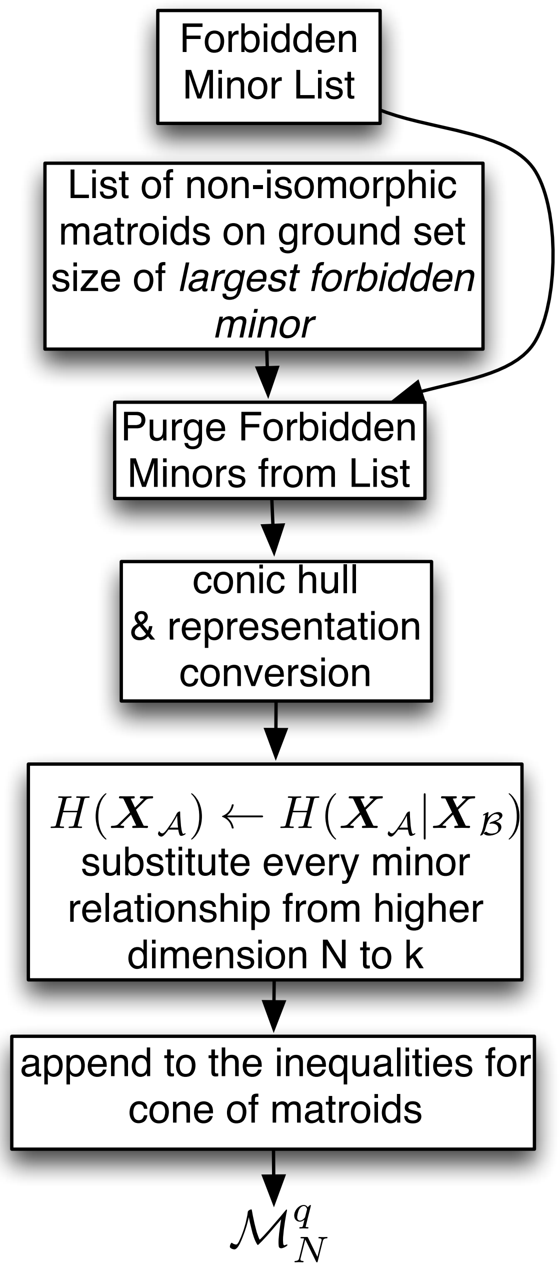
- All non-isomorphic matroids for $N \leq 9$ [8] '08
- enumerating non-iso. matroids is difficult

$GF(q)$-**Representable Matroid**: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$ s.t. $\exists \mathbf{A} \in GF(q)^{M\times N}$ s.t. $r(\mathcal{A}) = \operatorname{rank}(\mathbf{A}_{:,\mathcal{A}})$

- repr. matroid = scaled EV!: $\boldsymbol{u} \sim \mathcal{U}(GF(q)^M)$

$$\boldsymbol{X} = \boldsymbol{u}\mathbf{A} \;\Rightarrow\; h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

- Key: representability $\Leftrightarrow$ no forbidden minors:
  - *complete small list known for* $q \in \{2,3,4\}$ [9, 10, 11, 12, 13] eg.:$GF(2)$ repr. $\Leftrightarrow$ no $U(2,4)$ minor (*Tutte* 1958)
- $\bar{\Gamma}^*_N$ bound: $\mathcal{M}^q_N$ conic hull of $GF(q)$-repr. matroids. (*Hassibi* et. al. 2010 [14]). see right.



Bounding $\bar{\Gamma}^*_N$ from the Inside, 2: Inner Bounds from Subspace Ranks

Subspace Bounds: $\boldsymbol{r} \in \Gamma_N \cap \mathbb{Z}^{2^N-1}$ projections of representable matroids, $N' \geq N$, partition $\{1,\ldots,N'\} = \bigcup_{n=1}^{N} \mathcal{G}_n$, $\mathcal{G}_n \cap \mathcal{G}_k = \emptyset \; n \neq k$

$$r(\mathcal{A}) = \text{rank}([\mathbf{A}_{:,\mathcal{G}_n} | n \in \mathcal{A}]) \qquad (1)$$

subspace ranks = scaled EV!:

$$\boldsymbol{X}_n = \boldsymbol{u}\mathbf{A}_{:,\mathcal{G}_n} \Rightarrow h_{\mathcal{A}} = r(\mathcal{A}) \log_2 q$$

$\mathcal{S}_N$: conic hull of all subspace ranks

$\mathcal{S}_4$: $\Gamma_4 \cap$ Ingleton's [15, 16, 17]

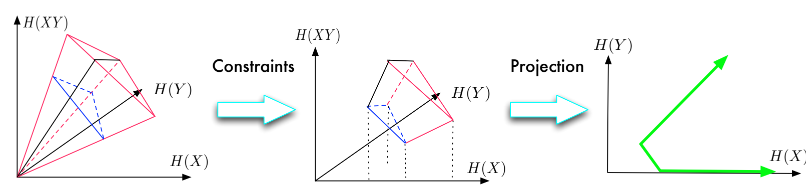
$$I(X_1; X_2) + I(X_3; X_4|X_1) + I(X_3; X_4|X_2) - I(X_3; X_4) \geq 0$$

$\mathcal{S}_5$ recently characterized by *DFZ* + *Kinser* [18, 19]

$\mathcal{S}_N$ unknown for $N \geq 6$, but can inner bounded by projecting $\mathcal{M}^q_N$ (see right)

Build inner bound for $\mathcal{S}_N \subsetneq \bar{\Gamma}^*_N$:

1. Obtain $\mathcal{M}^q_{N'}$ using method from previous slide, i.e. *intersect* Γ_N *with inequalities from forbidden minors & matroids*
2. project (remove all but entropies where each element in $\boldsymbol{X}_n$ appears together)



sound familiar???

Shannon → lin. $\bigcap$ → project

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Bounding $\bar{\Gamma}^*_N$ from the Inside, 3: Outer Bounds for Subspace Ranks $\mathcal{S}_N$

- Hammer et. al.: key [16] r.v.s X, Y made with subspaces: $\exists$ *common information* Z s.t.

$$H(Z|X) = H(Z|Y) = 0 \qquad (2)$$

$$H(Z) = I(X;Y) \qquad (3)$$

- Common information Z obtained by looking at component along intersection of subspaces X Y
- Not all RVs have common information, but rvs from subspaces do

<u>Build outer bound for $\mathcal{S}_N \subsetneq \bar{\Gamma}^*_N$:</u>

1. Shannon
2. Intersect with common information equalities (2)
3. project out the common informations Z

sound familiar?

Shannon $\rightarrow$ lin. $\bigcap$ $\rightarrow$ project

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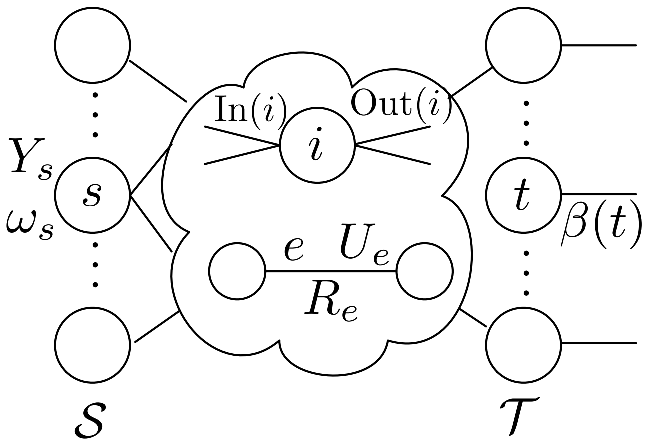
Networking Coding and Distributed Storage Rate Regions

Network Coding



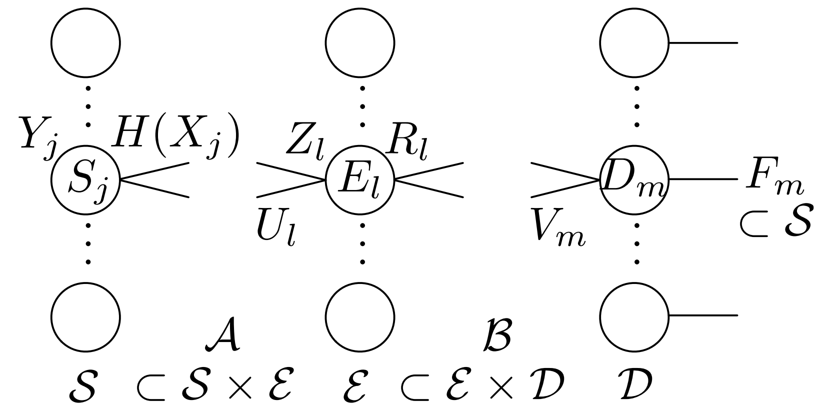
(Roughly) intersect $\bar{\Gamma}^*_N$ with

$$h_{Y_s} \geq \omega_s, \;\; s \in \mathcal{S} \qquad h_{Y_{\mathcal{S}}} = \sum_{s \in \mathcal{S}} h_{Y_s},$$

$$\begin{aligned} h_{U_{\mathrm{Out}(s)}|Y_s} &= 0, \;\; s \in \mathcal{S} \\ h_{U_{\mathrm{Out}(i)}|U_{\mathrm{In}(i)}} &= 0, \;\; i \in \mathcal{V} \setminus (\mathcal{S} \cup \mathcal{T}) \\ h_{U_e} &\leq R_e, \;\; e \in \mathcal{E} \\ h_{Y_{\beta(t)}|U_{\mathrm{In}(t)}} &= 0, \;\; t \in \mathcal{T} \end{aligned}$$

and project onto ω_s, R_e

Distributed Storage (MDCS DSCSC)



(Roughly) intersect $\bar{\Gamma}^*_N$ with

$$\begin{aligned} h_{Y_j, j \in \mathcal{S}} &= \sum_{j \in \mathcal{S}} h_{Y_j} \\ h_{Z_l|(Y_j, j \in U_l)} &= 0, \;\; l \in \mathcal{E} \\ h_{(Y_j, j \in F_m)|(Z_l, l \in V_m)} &= 0, \;\; m \in \mathcal{D} \\ h_{Y_j} &> H(X_j), \;\; j \in \mathcal{S} \\ h_{Z_l} &\leq R_l, \;\; l \in \mathcal{E} \end{aligned}$$

and project onto $\{H(X_i), R_l\}$

Substituting inner/outer bounds for $\bar{\Gamma}^*_N$, we arrive again at

Shannon $\rightarrow$ linear equality/inequality $\bigcap$ $\rightarrow$ project

One Procedure to Rule Four Problems

We've covered:

1. Non-Shannon Outer bounds for $\bar{\Gamma}^*_N$
2. Vector Matroidal Inner Bounds for $\bar{\Gamma}^*_N$
3. Outer Bound for $\mathcal{S}_N$ (Conic hull of subspace ranks)
4. Network Coding/Distributed Storage Rate Regions

(**Our Point**) There were $>$ four jokes... but only one punchline

$$\boxed{\text{Shannon} \rightarrow \text{linear equality/inequality} \bigcap \rightarrow \text{project}}$$

Thus, let's have a look at the general computational structure in this agenda.

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Many Paths Lead to the Same Truth

∃ wide variety of techniques (yielding mathematically equivalent results) for each step:

list of extreme rays

Shannon Outer Bound	list of inequalities		representation conversion		known list of non-isomorphic matroids
Adding linear inequalities/ equalities	append		representation conversion		run intermediate double descriptions step to update ray list with new inequalties
Project onto a subset of variables	Convex Hull Method	Fourier Motzkin	representation conversion	remove elements & convex hull	Extreme Point Method
Representation Conversion	Double Descriptions		Lexicographic Reverse Search	Exploit partial pairs/ lists?	Exploit Symmetries? sympol

Parallelization: message passing VS massively parallel GPU

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The Path Less Laden: Complexity Experiments

- mathematically equal $\neq$ computationally equal

Γ_N^{bin}	$N = 5$	6	7	8
Algorithm 2	1.3 s	11 s	150 s	3800 s
Algorithm 3	1.6 s	45 s	3000 s	36000 s
Algorithm 4	1.4 s	23 s	2000 s	25000 s

- Algorithm 2 uses substitution of conditional entropies into lower dimensional forbidden minor set to get inequality representation.
- Algorithm 3 & 4 work with non-isomorphic matroid list and remove minors (4: minor checking, 3: using 2's inequalities) to get extreme representation.

The Path Less Laden: Complexity Experiments

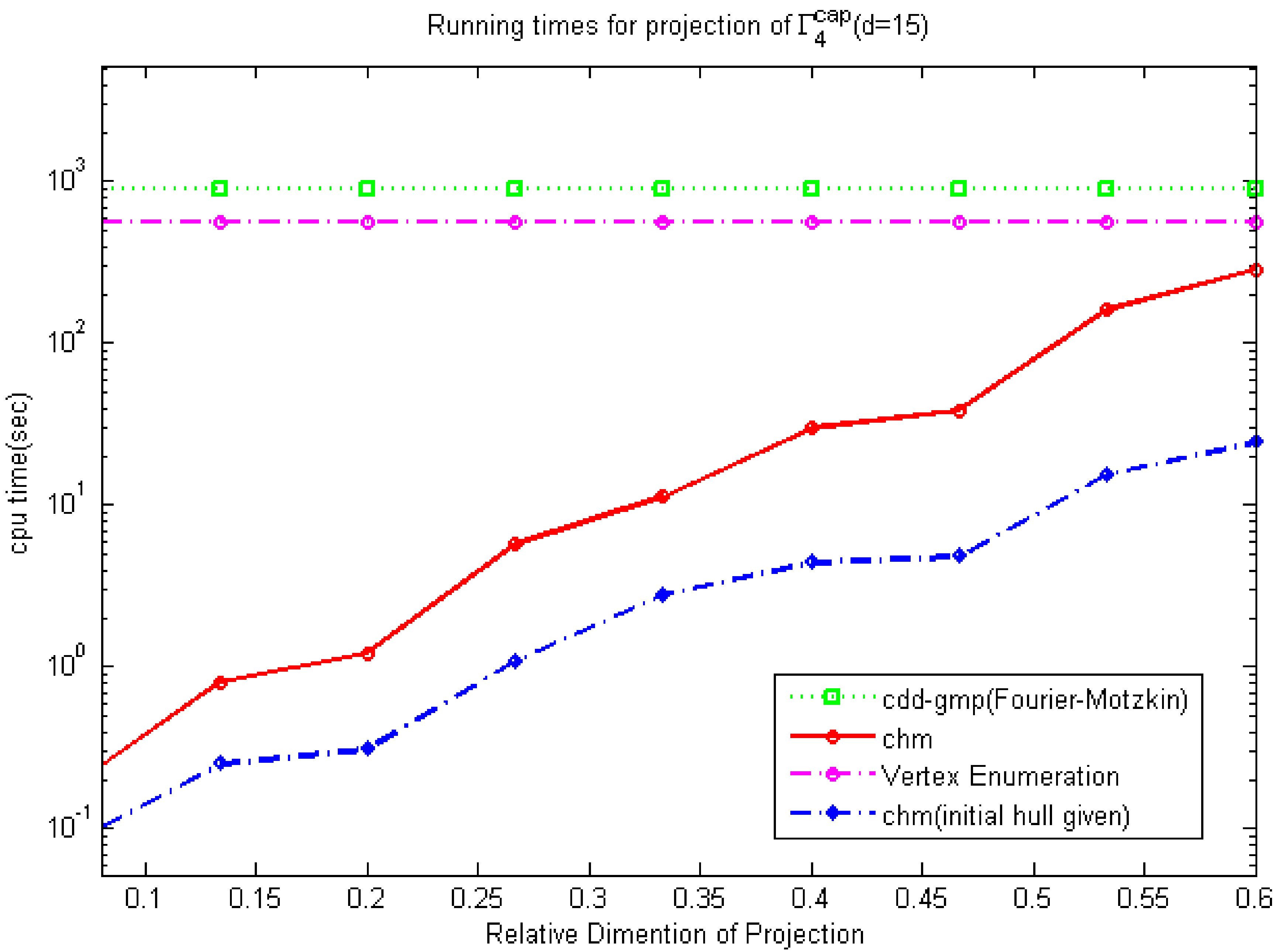
Rate Region	2-level-3-encoder	3-level-3-encoder
Algorithm 6	46 s	3600 s
Algorithm 7	2.9 s	47 s
Algorithm 8	2.7 s	35 s

- Algorithm 6: uses algorithm 2 to get inequalities of inner bound, appends rate region equalities/inequalities, and projects using Fourier Motzkin. (inequality based)

- Algorithm 7 & 8 : adds rate regions ineq.s & eq.s to alg. 3 and 4. inner bound extreme rays via steps of double descriptions, then projects

total time winners are not always concatenation of the winners at each stage

The Path Less Laden: Complexity Experiments



The Path Less Laden: Moving Forward

- There is an absurd amount of symmetry in these problems!
  - Labeling of variables, but also
  - Shannon is only one inequality! conditional mutual info ≥ 0
- Key factor in the computation is exploiting the known symmetry to ease the computation
- 2010 Diploma Thesis from Thomas Rehn Univ. Magdeburg (now at Uni. Rostock) – sympol
  - "representation conversion up to symmetries". D. Bremner & A. Schurmann
- Also, given that there is no unique "best path" and "best representation", yet a rich theory regarding which algorithms are good for which structures, need parallel tools that try the right candidates out then select which ones finish first
  - area of active software development (what our team is developing)
- given proof nature of results, need infinite precision arithmetic

The Path Less Laden: Moving Forward

What we envision:

- Polyhedral Computation Package
- Entropy Vector Bound Package
- Rate Region Package

Would you want to use this? Please tell me what you think about this, as well as any useful features I may have missed, after the talk.

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Structure of $\bar{\Gamma}_4^*$:

- ....Okay.... so, contradiction shows it's not polyhedral. (Matúš ISIT 2007.)
- But, how bad is it?
- In Matúš 1995 Conditional Independence Relations Paper:
  - series showed which Γ_4 faces have entropic point in relative interior, but also
  - Lemma 4: Gap between Shannon & Ingleton: 6 "pyramids" $\mathcal{P}_{ij} := \Gamma_4 \cap \{\text{Ingleton}_{ij} \leq 0\}$. Also, $\bar{\mathcal{P}}_{ij}^* := \bar{\Gamma}_4^* \cap \{\text{Ingleton}_{ij} \leq 0\}$
    - Only one of the 6 Ingletons can be violated at once

$$\Gamma_4 = \mathcal{I} \cup \cup_{ij} \mathcal{P}_{ij} \quad \bar{\Gamma}_4^* = \mathcal{I} \cup \cup_{ij} \bar{\mathcal{P}}_{ij}^* \tag{4}$$

    - Each of the Pyramids $\mathcal{P}_{ij}$: 1 non entropic extreme ray and 15 *binary* entropic extreme rays
- What can we infer about $\bar{\mathcal{P}}_4^* \cap \{\text{Ingleton}_{ij} \leq 0\}$ from these ingredients?

Structure of $\bar{\Gamma}_4^*$: Hey, maybe it's not so bad

h_1	h_2	h_{12}	h_3	h_{13}	h_{23}	h_{123}	h_4	h_{14}	h_{24}	h_{124}	h_{34}	h_{134}	h_{234}	h_{1234}
2	2	3	2	3	3	4	2	3	3	4	4	4	4	4
1	0	1	1	1	1	1	0	1	0	1	1	1	1	1
0	1	1	1	1	2	2	1	1	2	2	2	2	2	2
1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
0	1	1	1	1	1	1	0	0	1	1	1	1	1	1
1	0	1	1	2	1	2	1	2	1	2	2	2	2	2
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	2	1	2	2	3	1	2	2	3	2	3	3	3
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

h_1	h_2	h_{12}	h_3	h_{13}	h_{23}	h_{123}	h_4	h_{14}	h_{24}	h_{124}	h_{34}	h_{134}	h_{234}	h_{1234}
0	0	0	1	0	0	0	1	0	0	0	−1	0	0	0
−1	0	0	0	1	0	0	0	1	0	0	0	−1	0	0
−1	0	1	0	0	0	0	0	1	0	−1	0	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	0	0	0
0	−1	0	0	0	1	0	0	0	1	0	0	0	−1	0
0	−1	1	0	0	0	0	0	0	1	−1	0	0	0	0
0	−1	1	0	0	1	−1	0	0	0	0	0	0	0	0
0	0	0	−1	1	1	−1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	−1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	−1	1	1	−1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	−1	1	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1
1	1	−1	0	−1	−1	1	0	−1	−1	1	1	0	0	0

- Drop/Project out, e.g. $H(X_1)$ from $\mathcal{P}_{ij}$: the bad ray falls into the conic hull of the good ones $\Longrightarrow$ $\pi_{\setminus \mathcal{A}} \bar{\mathcal{P}}_4^* = \pi_{\setminus \mathcal{A}} \mathcal{P}_{ij}$
- happens if you drop any one of the 10 entropies $h_{\mathcal{A}} \in \mathrm{Ingleton}_{ij}$
- Dropping this entropy makes $\pi_{\setminus \mathcal{A}} \bar{\mathcal{P}}_{ij}^* = \{\mathbf{h}_{\setminus \mathcal{A}} | \mathbf{A}\mathbf{h}_{\setminus \mathcal{A}} \leq \mathbf{b}\}$ polyhedral

Structure of $\bar{\Gamma}_4^*$:

- Implication: for any $\mathcal{A}$ s.t. $h_{\mathcal{A}} \in \mathrm{Ingleton}_{ij}$, one way to express $\bar{\mathcal{P}}_{i,j}^*$ is

$$\bar{\mathcal{P}}_{i,j}^* = \left\{ \mathbf{h} \in \mathbb{R}^{15} \;\middle|\; \begin{array}{c} \mathbf{A}\mathbf{h}_{\setminus\mathcal{A}} \leq \mathbf{b} \ (= \mathrm{Shannon}) \\ h_{\mathcal{A}} \geq g_{\mathrm{low}}(\mathbf{h}_{\setminus\mathcal{A}}) \\ h_{\mathcal{A}} \leq g_{\mathrm{up}}(\mathbf{h}_{\setminus\mathcal{A}}) \end{array} \right\} \tag{5}$$

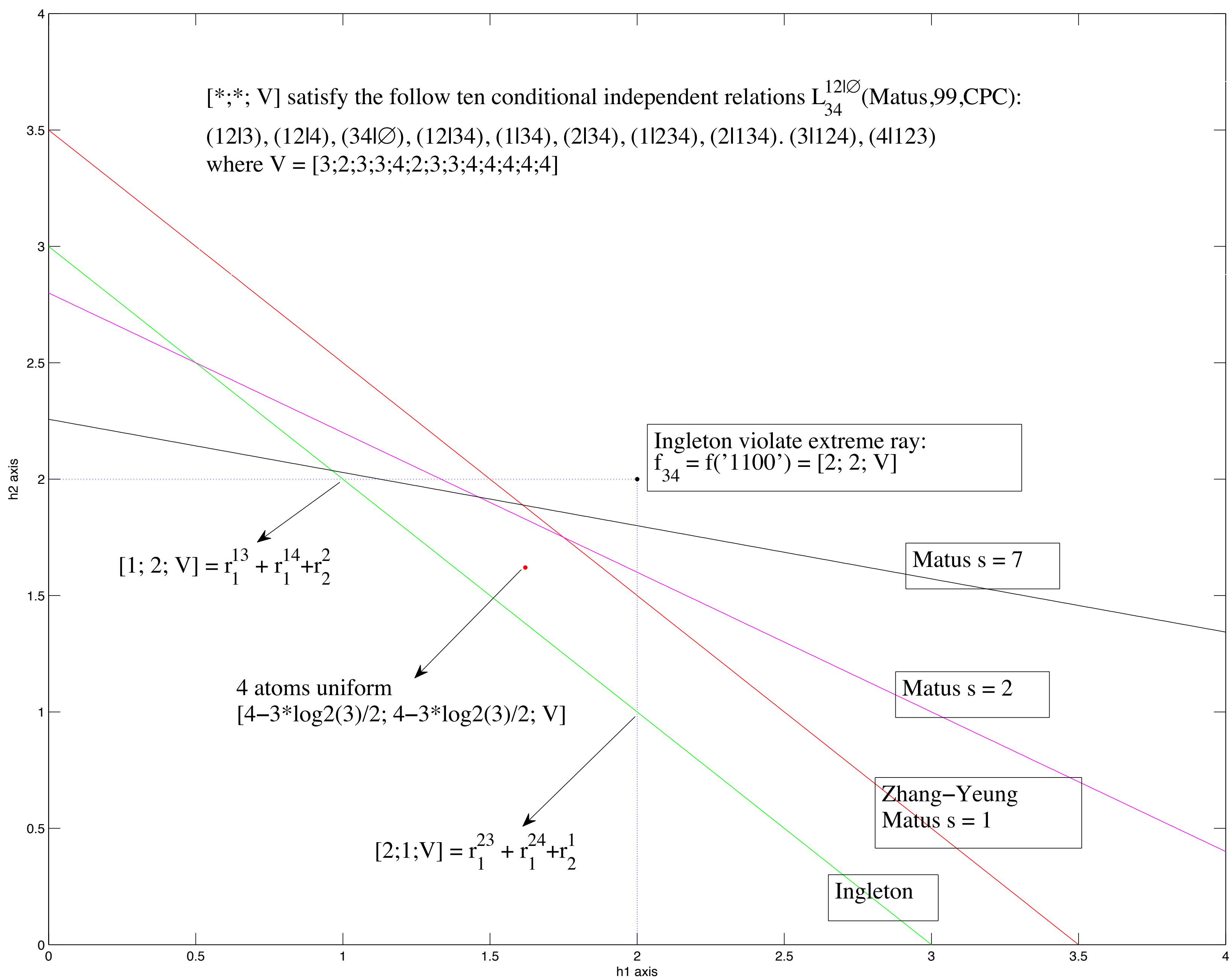
- Sign of $h_{\mathcal{A}} \in \mathrm{Ingleton}_{ij} \implies$ one of $g_{\mathrm{low}}(\mathbf{h}_{\setminus\mathcal{A}})$ or $g_{\mathrm{up}}(\mathbf{h}_{\setminus\mathcal{A}})$ from $\mathrm{Ingleton}_{ij} = 0$
- E.g. $\mathcal{A} = \{1\} \implies$
  $g_{\mathrm{low}}(\mathbf{h}_{\setminus 1}) = -h_2 + h_{12} + h_{23} + h_{13} - h_{123} + h_{14} + h_{24} - h_{124} - h_{34}$

The problem of determining $\bar{\Gamma}_4^*$ is equivalent, e.g., to determining a *single nonlinear function* $g_{\mathrm{up}} : \pi_{\setminus 1}\mathcal{P}_{12} \to \mathbb{R}_+$

$$\boxed{g_{\mathrm{up}}(\mathbf{h}_{\setminus 1}) := \max_{h_1 \;\mid\; \left[h_1, \mathbf{h}_{\setminus 1}^T\right]^T \in \Gamma_4^* \cap \{\mathrm{Ingleton}_{12} \leq 0\}} h_1}$$

(the solution to an optimization problem)

Structure of $\bar{\Gamma}_4^*$: Example



Structure of $\bar{\Gamma}_4^*$: Dropping h_{123}

h_1	h_2	h_{12}	h_3	h_{13}	h_{23}	h_{123}	h_4	h_{14}	h_{24}	h_{124}	h_{34}	h_{134}	h_{234}	h_{1234}
2	2	3	2	3	3	4	2	3	3	4	4	4	4	4
1	0	1	1	1	1	1	0	1	0	1	1	1	1	1
0	1	1	1	1	2	2	1	1	2	2	2	2	2	2
1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
0	1	1	0	0	1	1	1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	0	1	1	1	1	1	1	1
0	1	1	1	1	1	1	0	0	1	1	1	1	1	1
1	0	1	1	2	1	2	1	2	1	2	2	2	2	2
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	2	1	2	2	3	1	2	2	3	2	3	3	3
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

h_1	h_2	h_{12}	h_3	h_{13}	h_{23}	h_{123}	h_4	h_{14}	h_{24}	h_{124}	h_{34}	h_{134}	h_{234}	h_{1234}
0	0	0	1	0	0	0	1	0	0	0	−1	0	0	0
−1	0	0	0	1	0	0	0	1	0	0	0	−1	0	0
−1	0	1	0	0	0	0	0	1	0	−1	0	0	0	0
−1	0	1	0	1	0	−1	0	0	0	0	0	0	0	0
0	−1	0	0	0	1	0	0	0	1	0	0	0	−1	0
0	−1	1	0	0	0	0	0	0	1	−1	0	0	0	0
0	−1	1	0	0	1	−1	0	0	0	0	0	0	0	0
0	0	0	−1	1	1	−1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	−1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	−1	1	1	−1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	−1	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	−1	1	1	−1
0	0	0	0	0	0	0	0	0	0	0	0	−1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1
1	1	−1	0	−1	−1	1	0	−1	−1	1	1	0	0	0

$$\bar{\mathcal{P}}_{12}^* = \left\{ \begin{array}{c} \mathbf{A}\mathbf{h}_{\backslash 123} \leq \mathbf{b} \ (= \text{Shannon}) \\ h_{123} \geq g_{\text{low}}(\mathbf{h}_{\backslash 123}) \\ h_{123} \leq g_{\text{up}}(\mathbf{h}_{\backslash 123}) \end{array} \right\}$$

$$g_{\text{low}}(\mathbf{h}_{\backslash 123}) = -h_1 - h_2 + h_{12} + h_{23} + h_{13} + h_{14} + h_{24} - h_{124} - h_{34}$$

The problem of determining $\bar{\Gamma}_4^*$ is equivalent, e.g., to determining a single nonlinear function:

$g_{\text{up}} : \pi_{\backslash 123}\mathcal{P}_{12} \to \mathbb{R}_+$

$$g_{\text{up}}(\mathbf{h}_{\backslash 123}) := \max_{\left[h_{123}, \mathbf{h}_{\backslash 123}^T\right]^T \in \bar{\mathcal{P}}_{12}^*} h_{123}$$

E.g. Shannon says $g_{\text{up}}(\mathbf{h}_{\backslash 123}) \leq \min\{h_{2|1} + h_{13}, h_{2|3} + h_{13}, h_{1|2} + h_{23}, h_{1234}\}$

The lists of non-Shannon inequalities make the list of linear equations in the $\min$ larger.

The Case for Non-Polyhedral Tools:

Since $\bar{\Gamma}_N^*$ is a non-polyhedral convex cone:

- Need a tool that is not limited to (tightening of) polyhedral bounds
- Need a tool to handle non-linear codes!: $\mathcal{S}_N \subsetneq \bar{\Gamma}_N^* \; \forall N \geq 4$
  - Bye bye (linear) representable matroids. more general matroids promising path, but:
    - a pain to enumerate (list gigantic and unknown $N \geq 10$)
    - discrete $\implies$ conic hulls & rep. conv. nec. for REV. also expensive
    - algebraic matroids are far less understood than representable. other tools?

Extreme rays of $\bar{\Gamma}_N^*$ and $\cap$ correspond to efficient codes, hence:

- Want to parameterize the EVs and PMFs on boundary of $\bar{\Gamma}_N^*$
  - (esp. new extreme rays not shared with Shannon)

Information geometry:

- endows differential geometric structure to set of joint PMFs (parameterizations!)
- coord.'s flatness & certain affine sets = familiar properties
  - marginals, independence, conditional independence
- studies divergences (incl. KL) & projections that are easily related to entropy

Hence, information geometry seems a potential candidate to deal with these questions

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      ii. Vector Matroidal Inner bound for $\bar{\Gamma}_N^*$ (Hassibi)
      iii. Linear Code/Subspace Rank Outer Bound (Hammer, Kinser, DFZ)
      iv. Network Coding/Distributed Storage Regions (Yan, Yeung, Chan, Grant)
   (b) Many Paths Lead to the Same Truth
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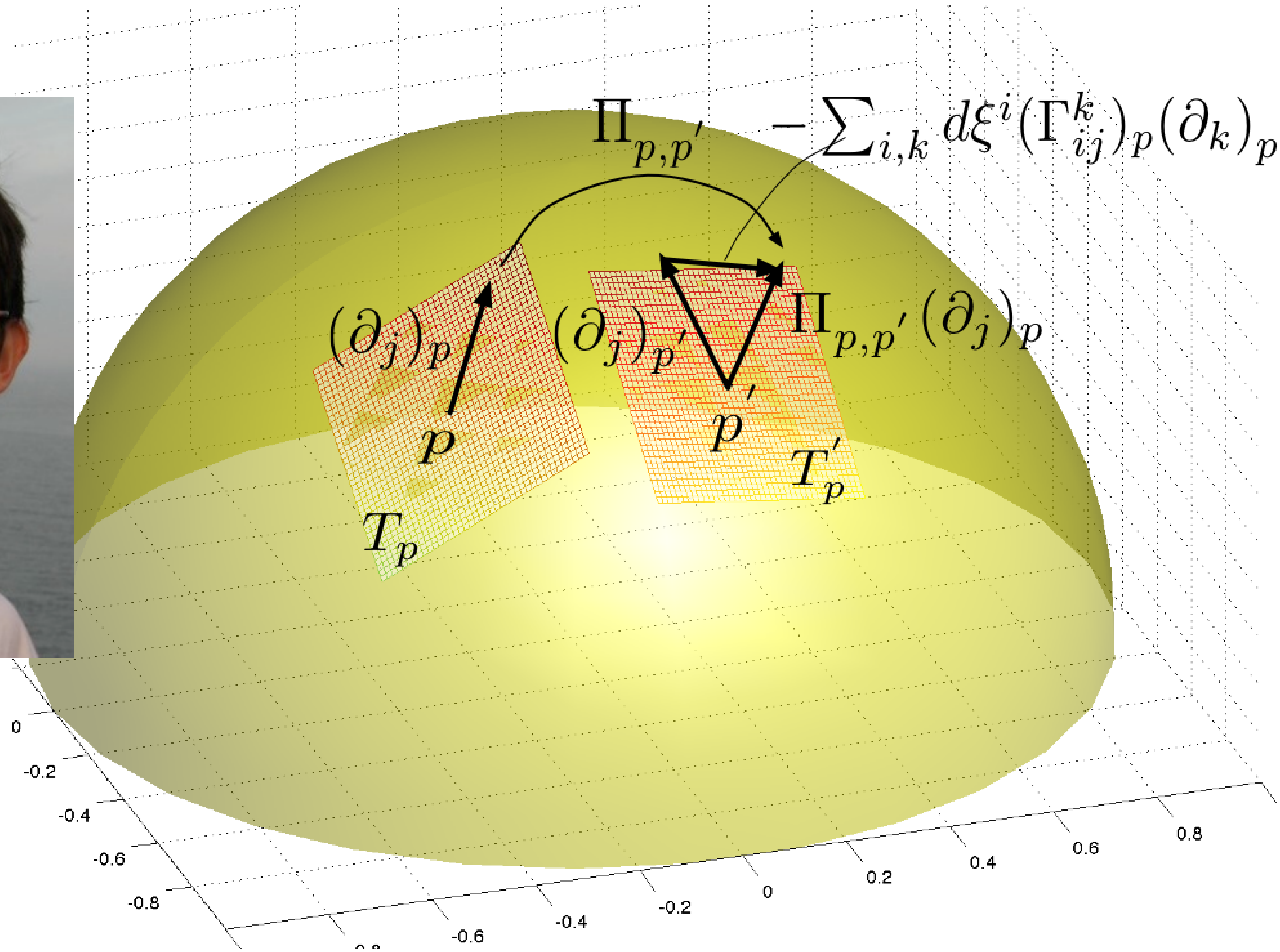
3. **Characterizing Extremal Entropic Vectors with Information Geometry**
   (a) Introduction to Information Geometry
   (b) Decoupling Constraints & Improving Ingleton w/ Binary Inner Bound
   (c) Casting Entropic Vectors as Information Projections
   (d) Information Geometric Properties of Distributions on Shannon Facets
   (e) $\bar{\Gamma}_4^*$ & 4 atom distribution. Information Geometry of Ingleton Violation

Characterizing Extremal Entropic Vectors with Information Geometry (Lead Student)

Yunshu Liu



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(Formal) Introduction to Information Geometry [20] – Notation

- Overall idea: treat family of probability distributions as a differentiable manifold: $p(x;\xi)$ is parameterized by ξ
- Endow w/ Riemannian metric (inner product between Tangent vectors) given by Fisher Information Matrix $g_{i,j}(\xi) = \mathbb{E}_\xi[\partial_i \ell_\xi \partial_j \ell_\xi]$ w/ $\ell_\xi = \log p(x;\xi)$, $\partial_i = \frac{\partial}{\partial \xi_i}$.
- Select α-affine connections $\nabla^{(\alpha)}$ such that $\left\langle \nabla^{(\alpha)}_{\partial_i} \partial_j, \partial_k \right\rangle = \Gamma^{(\alpha)}_{ij,k}$

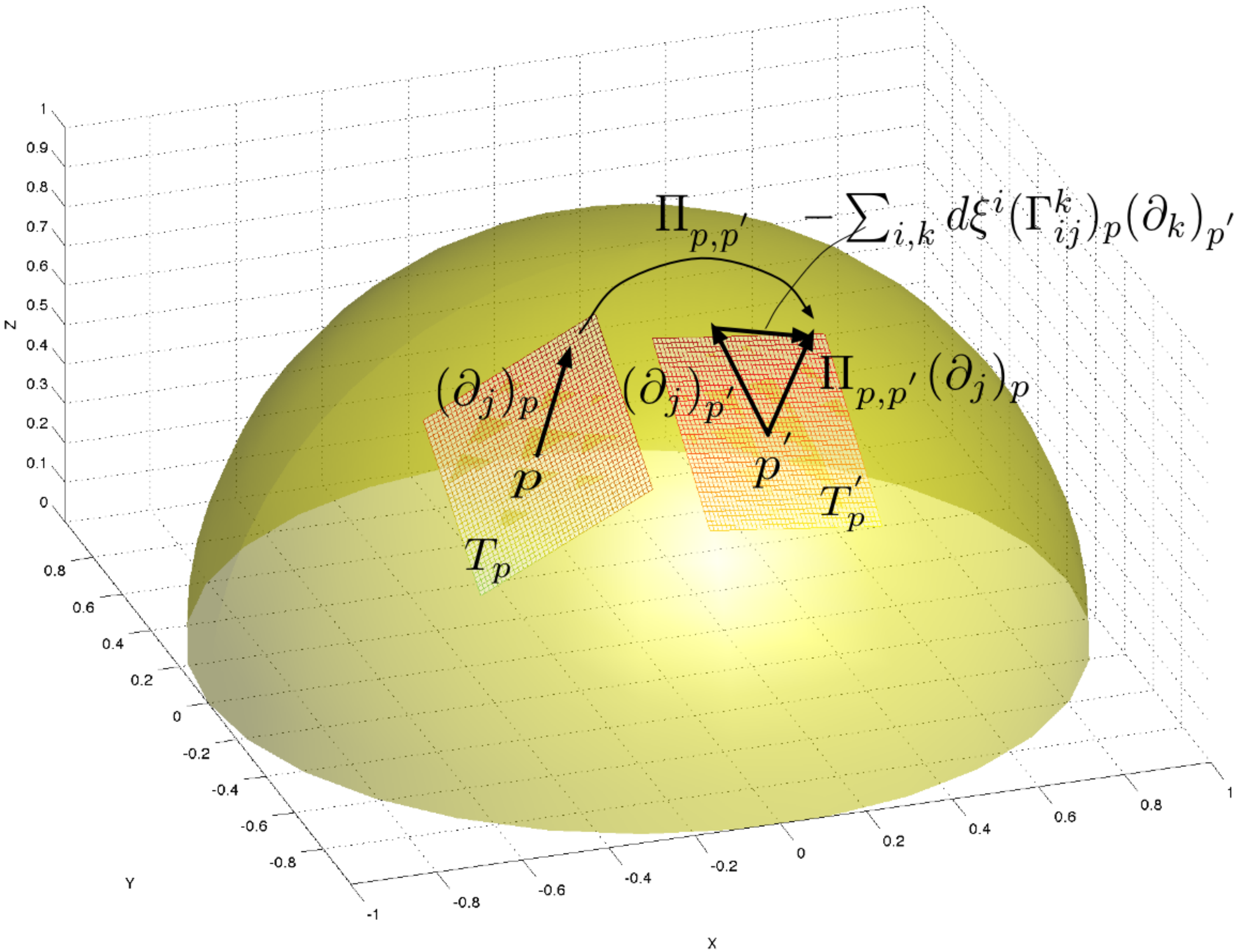
$$\Gamma^{(\alpha)}_{ij,k} = \mathbb{E}\left[\left(\partial_i \partial_j \ell_\xi + \frac{1-\alpha}{2} \partial_i \ell_\xi \partial_j \ell_\xi\right)(\partial_k \ell_\xi)\right] \tag{6}$$

- purpose of affine connection: define parallel translation $\Pi_{p,p'} : T_p \to T_{p'}$ to correspond tangent vectors along curves $\gamma : [a,b] \to \mathcal{P}$

$$\Pi_{\gamma(t),\gamma(t+dt)}(X(t)) = \sum_{ijk} \left\{ X^k(t) - dt \dot{\gamma}^i(t) X^j(t) \left(\Gamma_{ij,k}\right)_{\gamma(t)} \right\} (\partial_k)_{\gamma(t+dt)} \tag{7}$$

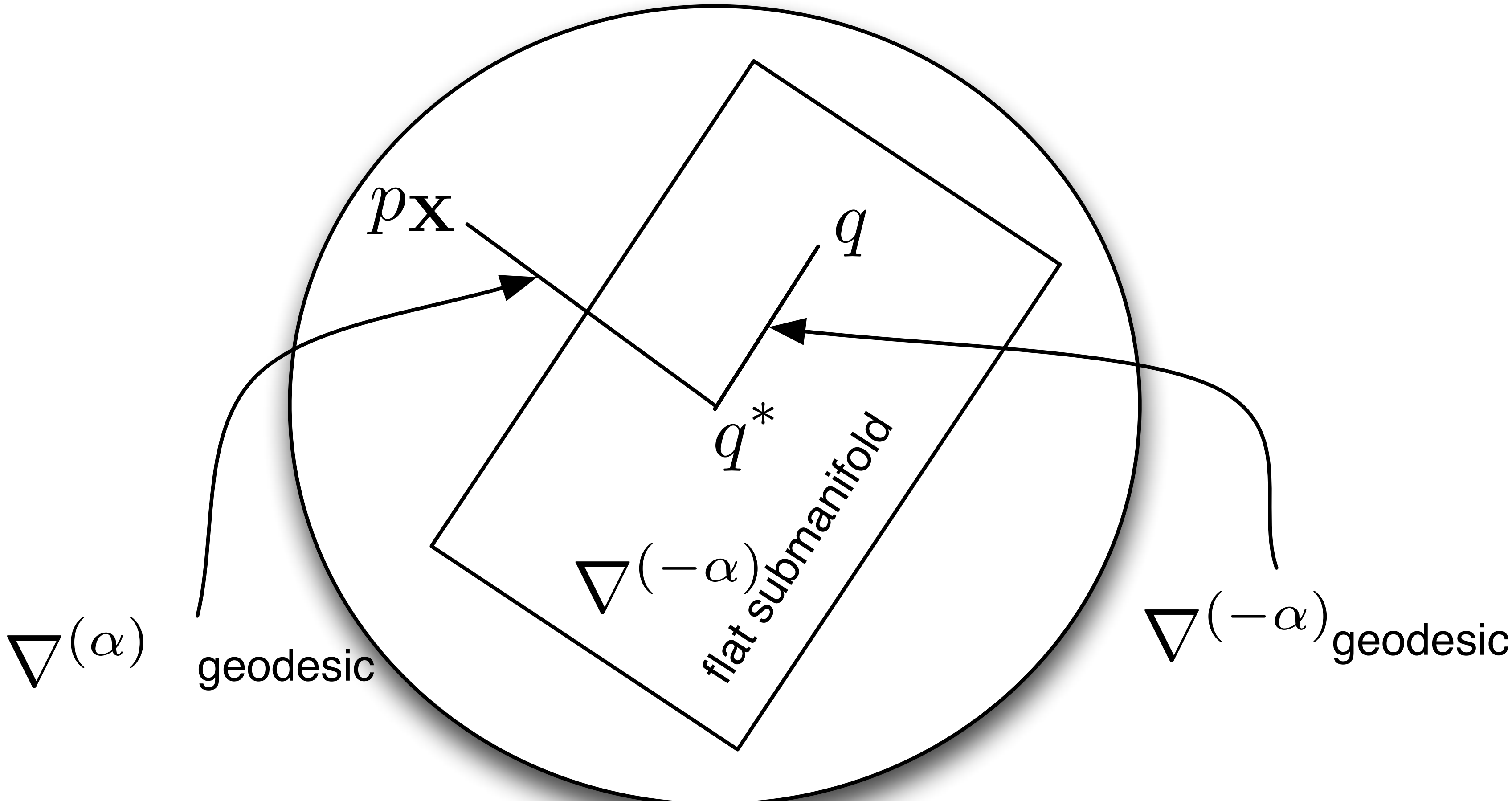
- Curve w/ tangent vector transported by parallel transl. w/ $\nabla^{(\alpha)}$ is $\nabla^{(\alpha)}$ *geodesic*
- If there is a coordinate system in which every parallel translation under $\nabla^{(\alpha)}$ leaves coefficients in Tangent vector unchanged, the manifold is said to be α-flat, and associated coordinate system is an affine coordinate system.
- $\nabla^{(\alpha)}$ has property $\langle X, Y \rangle_p = \langle \Pi^{(\alpha)}_{p,p'}(X), \Pi^{(-\alpha)}_{p,p'}(Y) \rangle_{p'}$

(Formal) Introduction to Information Geometry [20] – Parallel Translation



$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij,k}\partial_k \quad \Gamma_{ij,k} = 0 \text{ if "flat"} \tag{8}$$

(Formal) Introduction to Information Geometry [20] – Information Projection



$$D^{(\alpha)}(p_{\mathbf{x}}||q) = D^{(\alpha)}(p_{\mathbf{x}}||q^*) + D^{(\alpha)}(q^*||q)$$

(Informal) Introduction to Information Geometry [20] – Examples: Coordinates

m-coordinates:

$$\boldsymbol{\eta} = \left[p_{\mathbf{X}}(v_{i_1,1}, \ldots, v_{i_N,N}) \middle| \; i_k \in \{2, \ldots, |\mathcal{X}_k|\}, k \in \{1, \ldots, N\} \; \right]$$

e-coordinates: $\prod_{n=1}^{N} |\mathcal{X}_n| - 1$ elements take the form

$$\boldsymbol{\theta} = \left[\log\left(\frac{p_{\mathbf{X}}(v_{i_1,1}, \ldots, v_{i_N,N})}{p_{\mathbf{X}}(v_{1,1}, \ldots, v_{1,N})} \right) \middle| \begin{array}{c} i_k \in \{2, \ldots, |\mathcal{X}_k|\}, \\ k \in \{1, \ldots, N\} \end{array} \right]$$

m-autoparallel submanifold (affine subset of m-coords) fix $\mathbf{A}, \mathbf{b}$ all $\boldsymbol{\eta}$ of the form

$$\boldsymbol{\eta} = \mathbf{A}\boldsymbol{p} + \mathbf{b} \tag{9}$$

e-autoparallel submanifold (affine subset of e-coords) fix $\mathbf{A}, \mathbf{b}$ all $\boldsymbol{\theta}$ of the form

$$\boldsymbol{\theta} = \mathbf{A}\boldsymbol{\lambda} + \mathbf{b} \tag{10}$$

properties of affine sets $\implies$ intersections also affine, thus e/m affine closed under intersection.

e-geodesic/m-geodesic: one dimensional affine manifolds

(Informal) Introduction to Information Geometry [20] – Examples: Affine Sets

Examples of e-autoparallel submanifold:

- Set of joint distributions $p_{X,Y}$ s.t. X, Y indep.
- Set of joint distributions $p_{X,Y,Z}$ s.t. X,Y,Z indep. (etc)
- Set of joint distributions s.t. $X \leftrightarrow Y \leftrightarrow Z$

Examples of m-autoparallel submanifold

- Set of joint distributions $p_{X,Y}$ with a particular marginal distribution p_X
- Set of joint distributions $p_{X,Y}$ with a particular marginal distributions p_X, p_Y

(Informal) Introduction to Information Geometry [20] – Examples: Projections

- **e-flat submanifold:** set of all product distributions

$$\mathcal{E}_0 = \left\{ p_{\mathbf{X}} \middle| p_{\mathbf{X}}(x_1, \ldots, x_N) = \prod_{i=1}^{N} p_{X_i}(x_i) \right\} \tag{11}$$

- **m-flat submanifold:** set of joint distributions with given marginals

$$\mathcal{M}_0 = \left\{ p_{\mathbf{X}} \middle| \sum_{\mathbf{x}_{\setminus i}} p_{\mathbf{X}}(\mathbf{x}) = q_i(x_i) \quad \forall i \in \{1, \ldots, N\} \right\} \tag{12}$$

- **Information Projections & Pythagorean Relation:**

$$q^* = \arg\min_{q \in \mathcal{E}_0} D(p_{\mathbf{X}}||q), \quad D(p_{\mathbf{X}}||q) = D(p_{\mathbf{X}}||q^*) + D(q^*||q) \; \forall q \in \mathcal{E}_0 \tag{13}$$

$$q^* = \arg\min_{q \in \mathcal{M}_0} D(q||p_{\mathbf{X}}), \quad D(q||p_{\mathbf{X}}) = D(q^*||p_{\mathbf{X}}) + D(q||q^*) \; \forall q \in \mathcal{M}_0 \tag{14}$$

Information Geometry [20] – What has it been used for?

- re-interpretation of EM algorithm [20]
- acceleration of Blahut Arimoto algorithm [21]
- learning algorithms in Neural Networks [22]
- analysis of Belief propagation & Turbo Decoding [23, 24, 25, 26]

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Decoupling Constraints & Improving Ingleton w/ Binary Inner Bound

$$g_{\text{up}}(\mathbf{h}^o_{\setminus 123}) = \max_{\mathbf{h}\in\bar{\mathcal{P}}^*_{12} \big| \mathbf{h}_{\setminus 123} = \boldsymbol{h}^o_{\setminus 123}} h_{123} \textcolor{red}{\geq} \max_{\alpha_k \geq 0,\ \boldsymbol{\mathcal{X}},\ p^k_{\mathbf{X}} | \sum_k \alpha_k \mathbf{h}_{\setminus 123}(p^k_{\mathbf{X}}) = \mathbf{h}^o_{123}} \sum_k \alpha_k\ h_{123}(p^k_{\mathbf{X}})$$

$$= \max_{\alpha_k \geq 0,\ \boldsymbol{\mathcal{X}},\ \{p^k_{\mathbf{X}_{\mathcal{A}}} | \mathcal{A}\subset[4]\} \big| \sum_k \alpha_k H(p^k_{\mathbf{X}_{\mathcal{A}}}) = h^o_{\mathcal{A}},\ \sum_{\mathcal{A}^c} p^k_{\mathbf{X}} = p^k_{\mathbf{X}_{\mathcal{A}}} \forall \mathcal{A}\subset[4]} \sum_k \alpha_k H(p^k_{\mathbf{X}_{123}}) =$$

$$\max_{\substack{\alpha_k \geq 0, \boldsymbol{\mathcal{X}}, \\ p^k_{\mathbf{X}_{\mathcal{A}}} | \mathcal{A} \neq \{123\}, [4]\}}} \Bigg| \substack{\sum_k \alpha_k H(p^k_{\mathbf{X}_{\mathcal{A}}}) = h^o_{\mathcal{A}} \\ \exists p^k_{\mathbf{X}}, \sum_{\mathcal{A}^c} p^k_{\mathbf{X}} = p^k_{\mathbf{X}_{\mathcal{A}}}} \quad \max_{p^k_{\mathbf{X}_{123}}, p^k_{\mathbf{X}}} \Bigg| \substack{\sum_{\mathcal{A}^c} p^k_{\mathbf{X}} = p^k_{\mathbf{X}_{\mathcal{A}}}, \\ \sum_k \alpha_k H(p^k_{\mathbf{X}}) = h^o_{[4]}} \quad \sum_k \alpha_k H(p^k_{123})$$

(Think red term is actually equality. Matúš?) If we restrict domain to $\pi_{\setminus 123}\Phi_4$, restrict to a single fixed non-zero α $k = 1$ and $\boldsymbol{X} = \{0, 1\}$, then outer optimization has calculable finite # of points in feasible set [27]. Relies on handy m-affine re-parametrization that decouples the marginal constraints

$$q_{\mathcal{A}} = \mathbb{P}[\mathbf{X}_{\mathcal{A}} = \mathbf{1}_{|\mathcal{A}|}] \quad p_{\mathcal{A}}(\boldsymbol{x}_{\mathcal{A}}) = \sum_{\mathcal{C}|\mathcal{A}\subseteq\mathcal{C}\subseteq\mathcal{I}(\boldsymbol{x}_{\mathcal{A}})} (-1)^{|\mathcal{C}|-|\mathcal{I}(\boldsymbol{x}_{\mathcal{A}})|} q_{\mathcal{B}} \qquad (15)$$

which makes $h_{\mathcal{A}} = f(\{q_{\mathcal{B}} | \mathcal{B} \subseteq \mathcal{A}\})$ and turns solving inner optimization into only two parameter optimization problem: q_{123} and q_{1234}, i.e. can calculate this lower bound for g_{up}.

Decoupling Constraints & Improving Ingleton w/ Binary Inner Bound

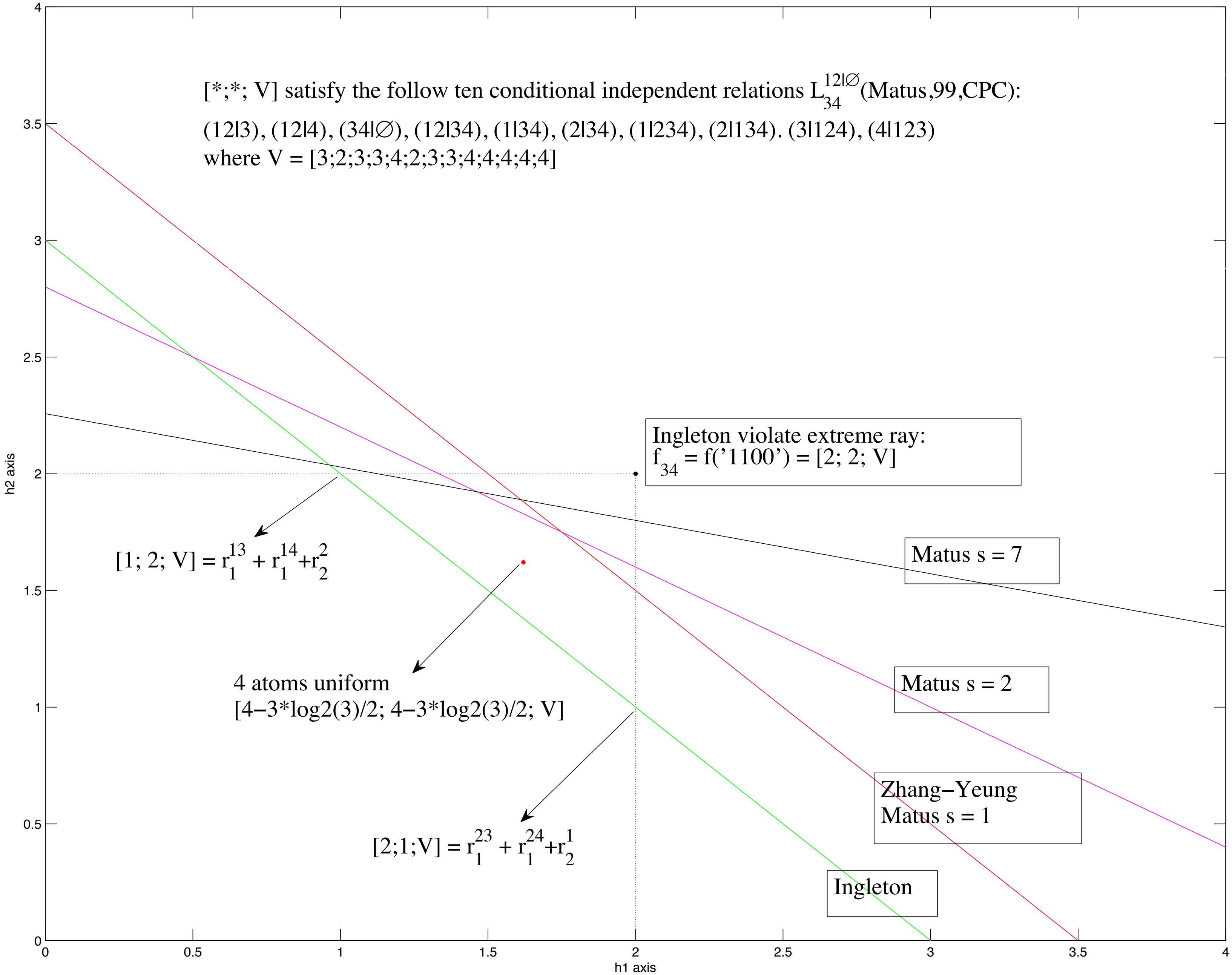
Why this may be a pretty good bound:

- All entropic extreme rays of $\mathcal{P}_{12}$ (Shannon rays on bottom of pyramid) are *binary*.
- Many/most Ingleton violating constructions have made use of non-unif binary r.v.s
- DFZ 4-atom conjecture about maximal Ingleton violation.

Moving forward

- The decoupling trick can be placed in an information geometric framework and generalized beyond binary.
- Inner optimization is almost convex (only one convex equality constraint is the problem). Just a little more transformation?

Structure of $\bar{\Gamma}_4^*$: Example



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Casting Entropic Vectors as Information Projections

Easy to relate Shannon entropy to rel. entropy/ KL Divergence:

$$
\begin{aligned}
D(p_{\mathbf{X}} || \mathcal{U}_{|\boldsymbol{\mathcal{X}}|}) &= \sum_{\mathbf{x} \in \boldsymbol{\mathcal{X}}} p_{\mathbf{X}}(\mathbf{x}) \log_2 \left(\frac{p_{\mathbf{X}}(\mathbf{x})}{1/|\boldsymbol{\mathcal{X}}|} \right) && (16) \\
&= \log_2(|\boldsymbol{\mathcal{X}}|) - H(p_{\mathbf{X}}) = H(\mathcal{U}_{\boldsymbol{\mathcal{X}}}) - H(p_{\mathbf{X}}) && (17)
\end{aligned}
$$

Casting Entropic Vectors as Information Projections

Next consider the family of distributions

$$\mathcal{H}_i := \left\{ p_{\mathbf{X}} \middle| p(\mathbf{X}) = \frac{1}{|\mathcal{X}_i|} q(\mathbf{X}_{\setminus i}), \text{ some } q(\mathbf{X}_{\setminus i}) \right\} \tag{18}$$

Observe:

- $\mathcal{U}_{\boldsymbol{\mathcal{X}}} \in \mathcal{H}_i$
- $\mathcal{H}_i$ is *both* an e-affine and m-affine submanifold.
- Defining $q^*_{\mathcal{H}_i}(p_{\mathbf{X}}) = \arg\min_{q \in \mathcal{H}_i} D(p_{\mathbf{X}} || q)$, have Pythagorean relation:

$$D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}}) = \underbrace{D(p_{\mathbf{X}} || q^*_{\mathcal{H}_i}(p_{\mathbf{X}}))}_{\log_2 |\mathcal{X}_i| - H(X_i | \mathbf{X}_{\setminus i})} + \underbrace{D(q^*_{\mathcal{H}_i}(p_{\mathbf{X}}) || \mathcal{U}_{\mathcal{X}})}_{\log_2 |\boldsymbol{\mathcal{X}}| - \log_2 |\mathcal{X}_i| - H(\mathbf{X}_{\setminus i})} \tag{19}$$

(erm... $H(\mathbf{X}) = H(X_i) + H(\mathbf{X}_{\setminus i} | X_i)$ tyco)

Moving this around, we have

$$H(\mathbf{X}_{\setminus i}) = D(p_{\mathbf{X}} || q^*_{\mathcal{H}_i}(p_{\mathbf{X}})) - D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}}) + \log_2 |\boldsymbol{\mathcal{X}}| - \log_2 |\mathcal{X}_i| \tag{20}$$

Casting Entropic Vectors as Information Projections

Generalizing this idea, consider the family of distributions

$$\bigcap_{i \in \mathcal{A}^c} \mathcal{H}_i = \left\{ p_{\mathbf{X}} = \frac{q(\mathbf{X}_{\mathcal{A}})}{\prod_{i \in \mathcal{A}^c} |\mathcal{X}_i|} \right\} \tag{21}$$

Observe:

- $\mathcal{U}_{\boldsymbol{\mathcal{X}}} \in \bigcap_{i \in \mathcal{A}^c} \mathcal{H}_i$
- $\bigcap_{i \in \mathcal{A}^c} \mathcal{H}_i$ is *both* an e-affine and m-affine submanifold
- Defining $q^*_{\mathcal{A}}(p_{\mathbf{X}}) = \arg\min_{q \in \bigcap_{i \in \mathcal{A}^c} \mathcal{H}_i} D(p_{\mathbf{X}} || q)$, have Pythagorean relation:

$$D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}}) = \underbrace{D(p_{\mathbf{X}} || q^*_{\mathcal{A}}(p_{\mathbf{X}}))}_{\sum_{i \in \mathcal{A}^c} \log_2 |\mathcal{X}_i| - H(\mathbf{X}_{\mathcal{A}^c} | \mathbf{X}_{\mathcal{A}})} + \underbrace{D(q^*_{\mathcal{A}}(p_{\mathbf{X}}) || \mathcal{U}_{\boldsymbol{\mathcal{X}}})}_{\log_2 |\boldsymbol{\mathcal{X}}| - \sum_{i \in \mathcal{A}^c} \log_2 |\mathcal{X}_i| - H(\mathbf{X}_{\mathcal{A}})} \tag{22}$$

(erm... $H(\mathbf{X}) = H(\mathbf{X}_{\mathcal{A}}) + H(\mathbf{X}_{\mathcal{A}^c} | \mathbf{X}_{\mathcal{A}})$ tyco)

From which we observe that

$$H(\mathbf{X}_{\mathcal{A}}) = D(p_{\mathbf{X}} || q^*_{\mathcal{A}}(p_{\mathbf{X}})) - D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}}) - \sum_{i \in \mathcal{A}^c} \log_2 |\mathcal{X}_i| + \log_2 |\boldsymbol{\mathcal{X}}| \tag{23}$$

Casting Entropic Vectors as Information Projections

Defining the set function (then stack into a vector $\mathbf{d}$)

$$d_{\mathcal{A}} := \min_{q \in \bigcap_{i \in \mathcal{A}^c} \mathcal{H}_i} D(p_{\mathbf{X}} || q) = D(p_{\mathbf{X}} || q_{\mathcal{A}}(p_{\mathbf{X}})) \quad \forall \mathcal{A} \subsetneq \{1, \ldots, N\} =: [N] \tag{24}$$

and $d_{\emptyset} = D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}})$. It is evident from the relation we derived

$$H(\mathbf{X}_{\mathcal{A}}) = D(p_{\mathbf{X}} || q^*_{\mathcal{A}}(p_{\mathbf{X}})) - D(p_{\mathbf{X}} || \mathcal{U}_{\boldsymbol{\mathcal{X}}}) - \sum_{i \in \mathcal{A}^c} \log_2 |\mathcal{X}_i| + \log_2 |\boldsymbol{\mathcal{X}}| \tag{25}$$

that

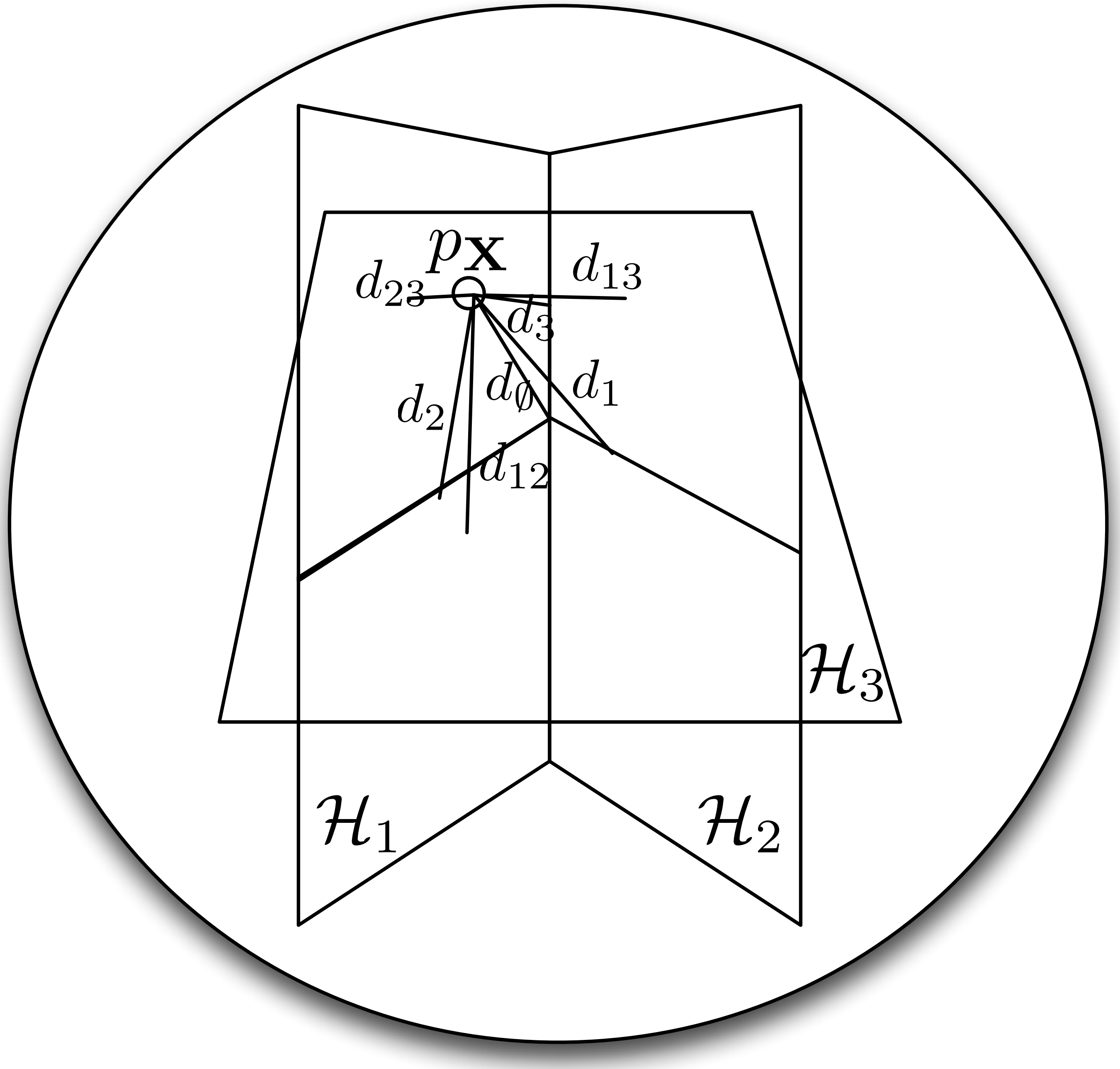
$$h_{\mathcal{A}} = d_{\mathcal{A}} - d_{\emptyset} - \sum_{i \in \mathcal{A}^c} \log_2 |\mathcal{X}_i| + \log_2 |\boldsymbol{\mathcal{X}}| \quad \forall \mathcal{A} \subsetneq [N] \tag{26}$$

and $h_{[N]} = -d_{\emptyset} + \log_2 |\boldsymbol{\mathcal{X}}|$, thus we can express entropic vector in terms of $\mathbf{d}$ via

$$\mathbf{h}(\mathbf{d}) = \mathbf{A}\mathbf{d} + \mathbf{b} \tag{27}$$

Region of entropic vectors is affine transformation of region of simultaneous divergences between submanifolds $\mathcal{H}_i$ and their intersections!

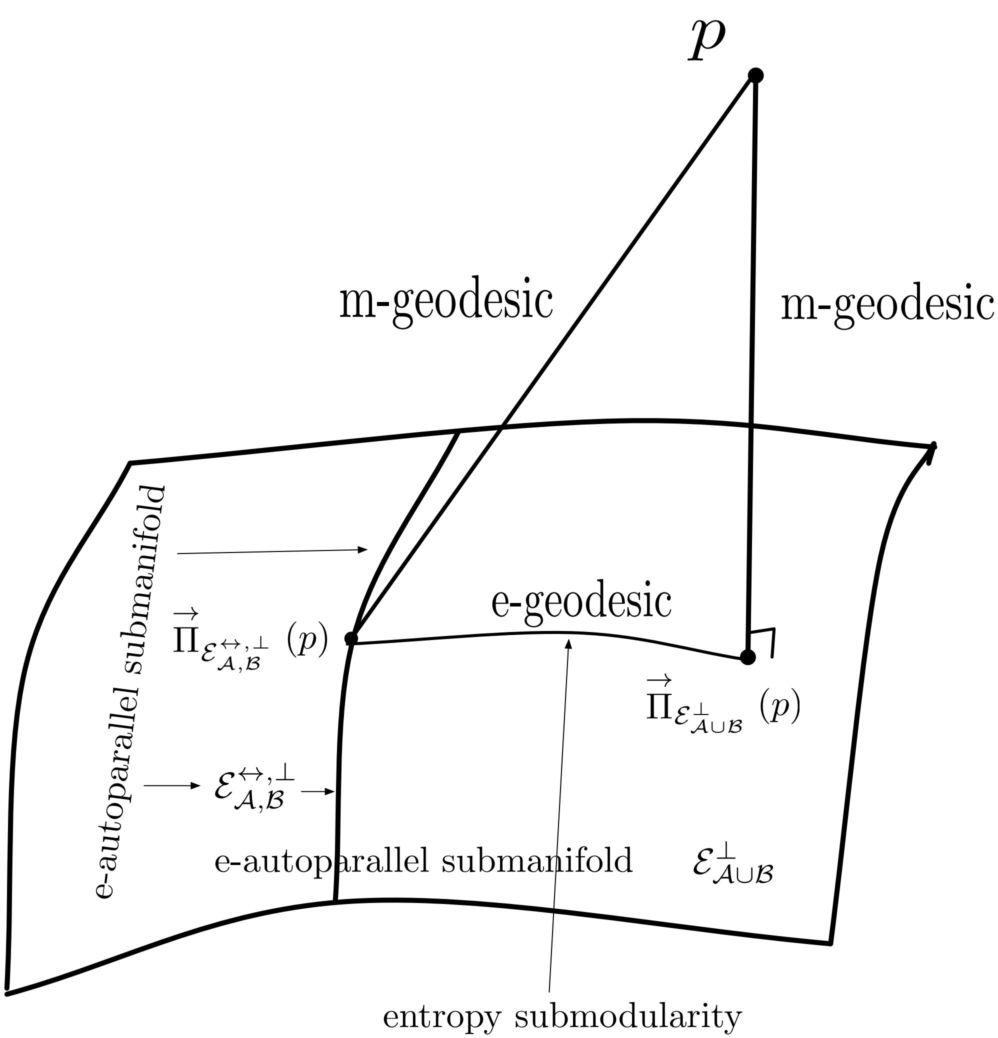
Casting Entropic Vectors as Information Projections



Outline

1. Entropic Vectors and Polyhedral Computation
   (a) One Procedure to Rule Four Problems
      i. Non-Shannon outer bounds for $\bar{\Gamma}_N^*$ (Yeung, Zhang, DFZ, Matúš, ...)
      ii. Vector Matroidal Inner bound for $\bar{\Gamma}_N^*$ (Hassibi)
      iii. Linear Code/Subspace Rank Outer Bound (Hammer, Kinser, DFZ)
      iv. Network Coding/Distributed Storage Regions (Yan, Yeung, Chan, Grant)
   (b) Many Paths Lead to the Same Truth
   (c) The Path Less Laden: Complexity Experiments & Moving Forward
2. Structure of $\bar{\Gamma}_4^*$ & the Case for Non-Polyhedral Tools
3. Characterizing Extremal Entropic Vectors with Information Geometry
   (a) Introduction to Information Geometry
   (b) Decoupling Constraints & Improving Ingleton w/ Binary Inner Bound
   (c) Casting Entropic Vectors as Information Projections
   (d) **Information Geometric Properties of Distributions on Shannon Facets**
   (e) $\bar{\Gamma}_4^*$ & 4 atom distribution. Information Geometry of Ingleton Violation

Information Geometric Properties of Distributions on Shannon Facets



$$h_{\mathcal{A}} + h_{\mathcal{B}} \geq h_{\mathcal{A}\cup\mathcal{B}} + h_{\mathcal{A}\cap\mathcal{B}}$$

$$\mathcal{E}^{\leftrightarrow,\perp}_{\mathcal{A},\mathcal{B}} = \left\{ \boldsymbol{\theta} \middle| p_{\boldsymbol{X}} = p_{\boldsymbol{X}_{\mathcal{A}\setminus\mathcal{B}}|\boldsymbol{X}_{\mathcal{A}\cap\mathcal{B}}} p_{\boldsymbol{X}_{\mathcal{B}}} p_{\boldsymbol{X}_{(\mathcal{A}\cup\mathcal{B})^c}} \right\}$$

$$\mathcal{E}^{\perp}_{\mathcal{A},\mathcal{B}} = \left\{ \boldsymbol{\theta} \middle| p_{\boldsymbol{X}} = p_{\boldsymbol{X}_{\mathcal{A}\cup\mathcal{B}}} p_{\boldsymbol{X}_{(\mathcal{A}\cup\mathcal{B})^c}} \right\}$$

- Shannon outer bound: $I(\boldsymbol{X}_{\mathcal{A}}; \boldsymbol{X}_{\mathcal{B}} | \boldsymbol{X}_{\mathcal{C}}) \geq 0$
- Hence, on the Shannon facet: $I(\boldsymbol{X}_{\mathcal{A}}; \boldsymbol{X}_{\mathcal{B}} | \boldsymbol{X}_{\mathcal{C}}) = 0$
- means $\boldsymbol{X}_{\mathcal{A}} \leftrightarrow \boldsymbol{X}_{\mathcal{C}} \leftrightarrow \boldsymbol{X}_{\mathcal{B}}$
- This is an e-autoparallel submanifold of $p_{\boldsymbol{X}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}}$!
- $\implies$ those $p_{\boldsymbol{X}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}}$ on this boundary (affine set) of entropy have a parameterization in which they are also affine (known $\mathbf{A}, \mathbf{b}$)
- Sometimes $\boldsymbol{X} \neq \boldsymbol{X}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}$, so also need the structure having a particular marginal $p_{\boldsymbol{X}_{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}}}$ (m-autoparallel)
- mutually dual foliations

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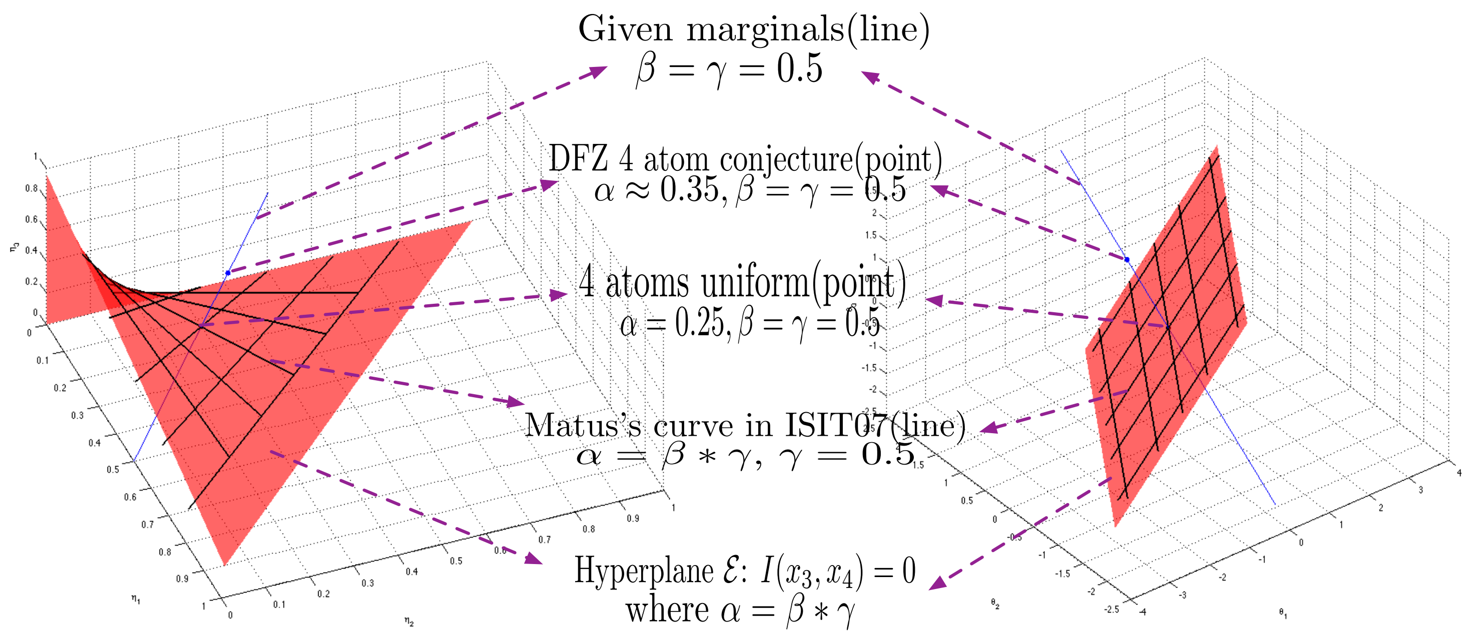
Left: (0000)(0110)(1010)(1111) in m-coordinate

Right: (0000)(0110)(1010)(1111) in e-coordinate



The whole 3D space

$p(0000) = \alpha$

$p(0110) = \beta - \alpha$

$p(1010) = \gamma - \alpha$

$p(1111) = 1 + \alpha - \gamma - \beta$

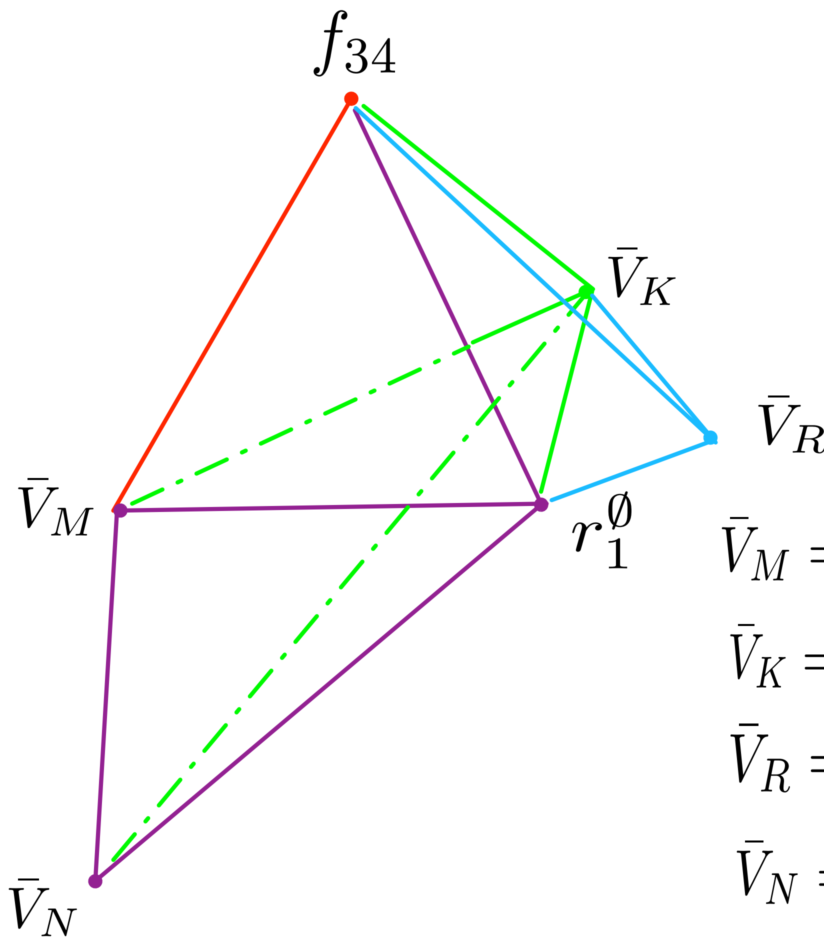
Marginal distribution of x_3 and x_4

$p(x_3 = 0) = \gamma$

$p(x_4 = 0) = \beta$

Structure of $\bar\Gamma_4^*$: Matúš Notation for $\mathcal{P}_{ij}$

$$\begin{aligned}\bar V_P &= (r_1^{\emptyset}, r_1^{3}, r_1^{4}, r_1^{13}, r_1^{14}, r_1^{23}, r_1^{24}, r_1^{123}, r_1^{124}, r_1^{134}, r_1^{234}, r_2^{1}, r_2^{2}, r_3^{\emptyset}, f_{34}) \\ &= (\bar V_M, \bar V_K, \bar V_R, r_1^{\emptyset}, f_{34})\end{aligned}$$

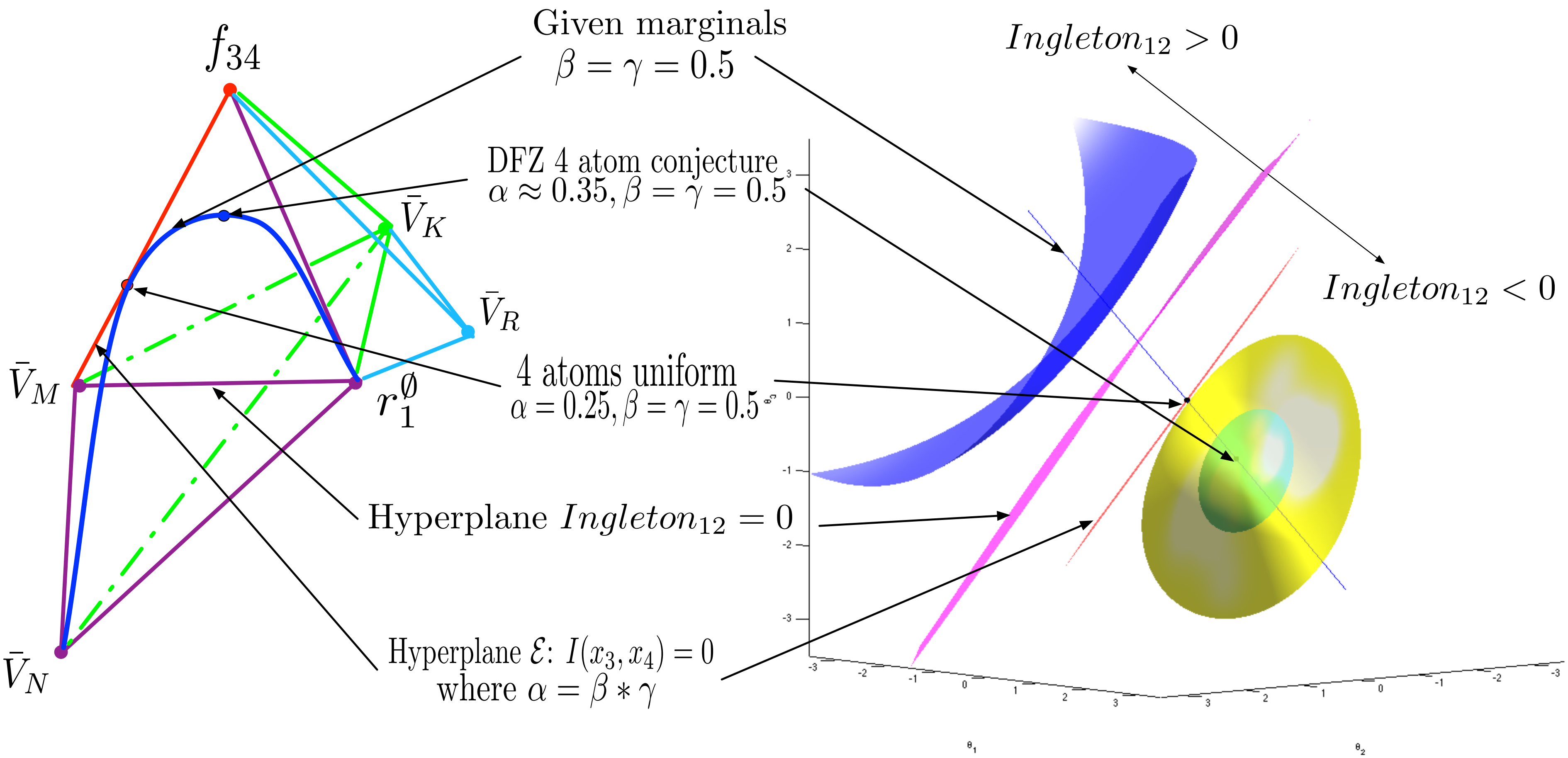


$$\bar V_M = (r_1^{13}, r_1^{14}, r_1^{23}, r_1^{24}, r_2^{1}, r_2^{2})$$

$$\bar V_K = (r_1^{123}, r_1^{124}, r_1^{134}, r_1^{234})$$

$$\bar V_R = (r_1^{3}, r_1^{4}, r_3^{\emptyset})$$

$$\bar V_N = (r_1^{1}, r_1^{2}, r_1^{12})$$



$$\bar{V}_P = (\bar{V}_M, \bar{V}_K, \bar{V}_R, r_1^{\emptyset}, f_{34})$$

where $\bar{V}_M = (r_1^{13}, r_1^{14}, r_1^{23}, r_1^{24}, r_2^{1}, r_2^{2})$ $\quad \bar{V}_K = (r_1^{123}, r_1^{124}, r_1^{134}, r_1^{234})$ $\quad \bar{V}_R = (r_1^{3}, r_1^{4}, r_3^{\emptyset})$ $\quad \bar{V}_N = (r_1^{1}, r_1^{2}, r_1^{12})$

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