Entropy and Information Inequalities Workshop
April 2013

Rearrangement and Entropy Inequalities

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Differential Entropy and Entropy Power

Let X be a random vector in $\mathbb{R}^n$ with probability density f. Then the differential entropy of X, $h(X)$, is defined as

$$h(X) = -\int f(x)\log(f(x))dx$$

The entropy power of X, $N(X)$, is defined as

$$N(X) = e^{\frac{2}{n}h(X)}$$

Remarks

- The differential entropy differs in many ways from discrete entropy. For example, $h(AX) = h(X) + \log(|A|)$
- The differential entropy can take all values in $[-\infty, +\infty]$. Hence $N(X)$ can assume any values in $[0, +\infty]$
- If X is an isotropic Gaussian N$(0, \sigma^2\mathbb{I})$, then $N(X) = 2\pi e\sigma^2$

Original Entropy Power Inequality

The Entropy Power Inequality (EPI) of [Shannon '48, Stam '59] is as follows:

If X_1 and X_2 be two independent random vectors in $\mathbb{R}^n$:

$$e^{\frac{2h(X_1+X_2)}{n}} \geq e^{\frac{2h(X_1)}{n}} + e^{\frac{2h(X_2)}{n}}$$

Many motivations

- Useful in proving converses in coding theorems
- Implies, for i.i.d. X_i,

$$h\left(\frac{X_1 + X_2}{\sqrt{2}}\right) \geq h(X_1)$$

  and hence is a key step to the entropic Central Limit Theorem
- Implies the Heisenberg Uncertainty Principle

Equivalent formulation of EPI

2nd Formulation If X_1, X_2 are independent random vectors in $\mathbb{R}^n$ and Z_1, Z_2 are independent isotropic Gaussian random vectors in $\mathbb{R}^n$ such that $h(X_1) = h(Z_1), h(X_2) = h(Z_2)$,

$$h(X_1 + X_2) \geq h(Z_1 + Z_2)$$

with equality iff X_1 and X_2 are normal with proportional covariance matrices

Why? Let Z_1 and Z_2 be two independent, isotropic (i.e., with covariance matrix a multiple of identity) Gaussian random vectors in $\mathbb{R}^n$ such that

$$h(X_1) = h(Z_1), \qquad h(X_2) = h(Z_2)$$

Then, assuming the original formulation,

$$e^{\frac{2h(X_1+X_2)}{n}} \geq e^{2h(X_1)/n} + e^{2h(X_2)/n} = e^{2h(Z_1)/n} + e^{2h(Z_2)/n} = e^{2h(Z_1+Z_2)/n}$$

Conversely, assuming the second formulation,

$$e^{\frac{2h(X_1+X_2)}{n}} \geq e^{2h(Z_1+Z_2)/n} = e^{2h(Z_1)/n} + e^{2h(Z_2)/n} = e^{2h(X_1)/n} + e^{2h(X_2)/n}$$

Digression: An Open Problem

We define the Stam region to be the subset of $\mathbb{R}^{2^k-1}$ given by

$$\Gamma(k) = \{[N\Big(\sum_{i\in s} X_i\Big)]_{s\neq\emptyset\subset[k]} : X_1, X_2, \ldots, X_k$$

are independent random vectors (in some $\mathbb{R}^n$)}

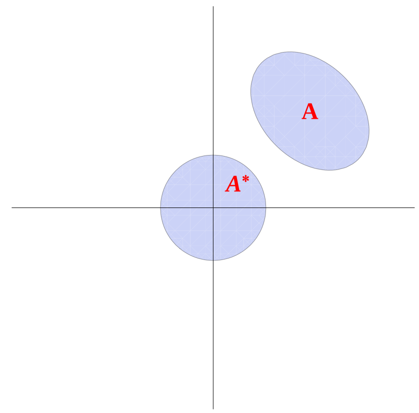
Question Can we characterize the Stam region?

Known Results

- It was shown in [Madiman and Barron '07] that the set function $v(s) = N\Big(\sum_{i\in s} X_i\Big)$ is fractionally superadditive
- We can show that the set function $v(s)$ is NOT always supermodular

A Brief Introduction to Rearrangements

The rearrangement of a set $A \subset \mathbb{R}^n$ is just the Euclidean ball with the same volume as A centered at 0 and is denoted A^*



For a function $f : \mathbb{R}^n \to [0, \infty)$, let $A_t = \{x : f(x) \geq t\}$. Define

$$f^*(x) = \int_0^\infty 1_{\{x \in {A_t}^*\}} dt$$

where $1_{\{x \in {A_t}^*\}} = 1$ if $x \in {A_t}^*$ and 0 otherwise. We call f^* the spherically symmetric decreasing rearrangement of f

Remarks

- Note that $f(x) = \int_0^{f(x)} dt = \int_0^\infty 1_{\{f(x) \geq t\}} dt = \int_0^\infty 1_{\{x \in A_t\}} dt$
- So the idea of the definition is to build up f^* from the rearranged super-level sets in the same way that we can build f from its super-level sets

Discrete Analogues of f^*

Consider a probability mass function $p(i)$ on $\mathbb{Z}$. Let us define p^+ to be a permutation of p

$$p^+(0) \geq p^+(1) \geq p^+(-1) \geq p^+(2) \geq p^+(-2)$$
$$\geq \cdot\cdot \geq p^+(i) \geq p^+(-i) \cdot\cdot$$

and ${}^+p$ to be a permutation of p

$${}^+p(0) \geq {}^+p(-1) \geq {}^+p(1) \geq {}^+p(-2) \geq {}^+p(2)$$
$$\geq \cdot\cdot \geq {}^+p(-i) \geq {}^+p(i) \cdot\cdot$$

If $p^+ = {}^+p$, we define

$$p^* = p^+$$

Note that p^* is symmetric and decreasing on the set of non-negative integers. This is precisely the discrete analogue of f^*

Example

Let $f(x) = \frac{1}{\sqrt{(2\pi)^n|A|}} e^{-\frac{x^T A^{-1} x}{2}}$ be a non-degenerate Gaussian with covariance matrix A. Then

$$A_t = \{y : f(y) > t\} = \{y : y^T A^{-1} y < -2\log(t\sqrt{(2\pi)^n|A|})\}$$

if $t < \frac{1}{\sqrt{(2\pi)^n|A|}}$ and empty otherwise

$$A_t^* = \{y : y^T y < -2|A|^{\frac{1}{n}} \log(t\sqrt{(2\pi)^n|A|})\}$$

if $t < \frac{1}{\sqrt{(2\pi)^n|A|}}$ and empty otherwise. Recall that:

$$f^*(x) = \int_0^\infty 1_{\{x \in A_t{}^*\}} dt$$

But $1_{\{x \in A_t{}^*\}} = 1$ if and only if $t < \frac{1}{\sqrt{(2\pi)^n|A|}} e^{-\frac{x^T x}{2|A|^{\frac{1}{n}}}}$. Hence finally,

$$f^*(x) = \frac{1}{\sqrt{(2\pi)^n|A|}} e^{-\frac{x^T x}{2|A|^{\frac{1}{n}}}}$$

Some Lemmas

Lemma 1 For any function $f : \mathbb{R}^n \to [0, \infty)$, and all $t \geq 0$,

$$\{x : f(x) > t\}^* = \{x : f^*(x) > t\}$$

Lemma 2 (Rearrangement preserves densities) For any function $f : \mathbb{R}^n \to [0, \infty)$, and any $p \geq 1$,

$$\|f\|_p = \|f^*\|_p$$

In particular, the rearrangement of a density function is a density function

Lemma 3 (Rearrangement preserves entropy) For any random vector X with density f,

$$h(X) = h(X^*)$$

where X^* is distributed according to f^*

Remark

Lemma 1 implies that f^* is a spherically symmetric decreasing function (i.e., $f^*(x)$ only depends on $|x|$ and is non-increasing in it) such that, for any measurable subset $B \subset [0, \infty)$, the volumes of the sets $\{x : f(x) \in B\}$ and $\{x : f^*(x) \in B\}$ are the same

Main Result: An EPI with Rearrangements

Suppose X_1 and X_2 are two independent random vectors in $\mathbb{R}^n$ with densities f_1 and f_2. Then

$$h(X_1 + X_2) \geq h(X_1^* + X_2^*)$$

where X_1^* and X_2^* are independent with densities f_1^* and f_2^*

Remarks

- Since $h(X_i^*) = h(X_i)$, the 2nd formulation of EPI applied to X_1^* and X_2^* implies

  $$h(X_1^* + X_2^*) \geq h(Z_1 + Z_2)$$

  so that we can write

  $$h(X_1 + X_2) \geq h(X_1^* + X_2^*) \geq h(Z_1 + Z_2)$$

  Hence it can indeed be seen as a kind of strengthening

- However, note that this does not directly give a new proof of the EPI since we used the EPI to show that it was a strengthening!

Rényi Entropy of Order p

Definition

$$h_p(X) = \frac{1}{1-p} \log \int f^p(x)dx$$

where $p \in (0,1) \cup (1, +\infty)$

Remarks

- $h_1(X)$ is defined as the Shannon differential entropy

$$h(X) = h_1(X) = -\int_{\mathbb{R}^n} f(x) \log f(x) dx$$

- As $p \to 0$, $h_p(X)$ reduces to

$$h_0(X) = \log |\text{Supp}(f)|$$

where $\text{Supp}(f) = \{x : f(x) > 0\}$ is the support of the density f

- As $p \to +\infty$, $h_p(X)$ reduces to

$$h_\infty(X) = -\log(\text{esssup}_x f(x))$$

where $\text{esssup}_x f(x)$ is the essential supremum of f

A More General Result

Suppose X_1 and X_2 are two independent random vectors in $\mathbb{R}^n$ with densities f_1 and f_2. Then

$$h_p(X_1 + X_2) \geq h_p(X_1^* + X_2^*)$$

where X_1^* and X_2^* are independent with densities f_1^* and f_2^* and $p \in [0, +\infty]$

Remarks

- In fact, we proved the following, which is even stronger

$$\int \phi(f_1 \star f_2 \star \cdot \cdot \star f_k)(x)dx \leq \int \phi(f_1^* \star f_2^* \star \cdot \cdot f_k^*)(x)dx$$

  where ϕ is a continuous convex function on $[0, \infty)$ such that $\phi(0) = 0$

- By taking $\phi = x^p$ for $p > 1$, $\phi = x \log(x)$ for $p = 1$ and $\phi = -x^p$ for $p < 1$, we obtain

$$h_p(X_1 + X_2 + \cdot \cdot + X_k) \geq h_p(X_1^* + X_2^* + \cdot \cdot + X_k^*)$$

- The proof is based on the generalized Riesz Sobolev convolution inequality and ideas from majorization theory

Application 1

Question Can we get a new proof of the Original EPI from our Main Result?

Special Case: Assume X_1 and X_2 are IID

$$h\left(\frac{X_1 + X_2}{\sqrt{2}}\right) \geq h(X_1)$$

Proof

- It suffices to show

$$h\left(\frac{X_1^* + X_2^*}{\sqrt{2}}\right) \geq h(X_1^*)$$

since then by our main result and Lemma 3 (Rearrangement preserves entropy)

$$h\left(\frac{X_1 + X_2}{\sqrt{2}}\right) \geq h\left(\frac{X_1^* + X_2^*}{\sqrt{2}}\right) \geq h(X_1^*) = h(X_1)$$

- By independence, $h(X_1^*, X_2^*) = h(X_1^*) + h(X_2^*)$. By spherical symmetry and i.i.d. assumption

$$\frac{X_1^* + X_2^*}{\sqrt{2}} =^d \frac{X_1^* - X_2^*}{\sqrt{2}}$$

By the scaling property for entropy and subadditivity

$$h(X_1^*, X_2^*) = h\left(\frac{X_1^* + X_2^*}{\sqrt{2}}, \frac{X_1^* - X_2^*}{\sqrt{2}}\right) \leq h\left(\frac{X_1^* + X_2^*}{\sqrt{2}}\right) + h\left(\frac{X_1^* - X_2^*}{\sqrt{2}}\right)$$

Hence

$$h\left(\frac{X_1^* + X_2^*}{\sqrt{2}}\right) \geq h(X_1^*)$$

Application 2

Brunn-Minkowski inequality

Let A, B be any Borel sets in $\mathbb{R}^n$. Write $A+B = \{x+y : x \in A,\ y \in B\}$ for the Minkowski sum, and $|A|$ for the n-dimensional volume. Then

$$\left|A+B\right|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} \quad [BM]$$

Proof

BM follows from $p = 0$ case of our more general result. The Brunn-Minkowski inequality can be rewritten as

$$\left|A+B\right|^{\frac{1}{n}} \geq \left|A^* + B^*\right|^{\frac{1}{n}}$$

But the $p = 0$ case of our strengthened result implies

$$\left|A+B\right| \geq \left|\text{supp}(X+Y)\right| \geq \left|\text{supp}(X^* + Y^*)\right| = \left|A^* + B^*\right|$$

where all random vectors are independent and X and Y are uniform distributions on A and B

Summary

- A New EPI refining the classical one
- May be useful in reducing information-theoretic problems to the special case involving spherically symmetric, decreasing densities, for which the symmetry can be used
- Recovers several classical inequalities as corollaries

Thank you!

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