How did we get into this mess?

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1988

R. W. Yeung, “Some result on multiterminal source coding,” PhD thesis, Cornell University, 1988

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DISSERTATION

SOME RESULTS ON MULTITERMINAL SOURCE CODING

RAYMOND WAI HO YEUNG

1988

1991

R. W. Yeung, "A new outlook on Shannon's information measures," IEEE-IT, 1991

Shannon’s Information Measures

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- In information theory, entropy is the measure of the uncertainty contained in a discrete random variable, justified by fundamental coding theorems.

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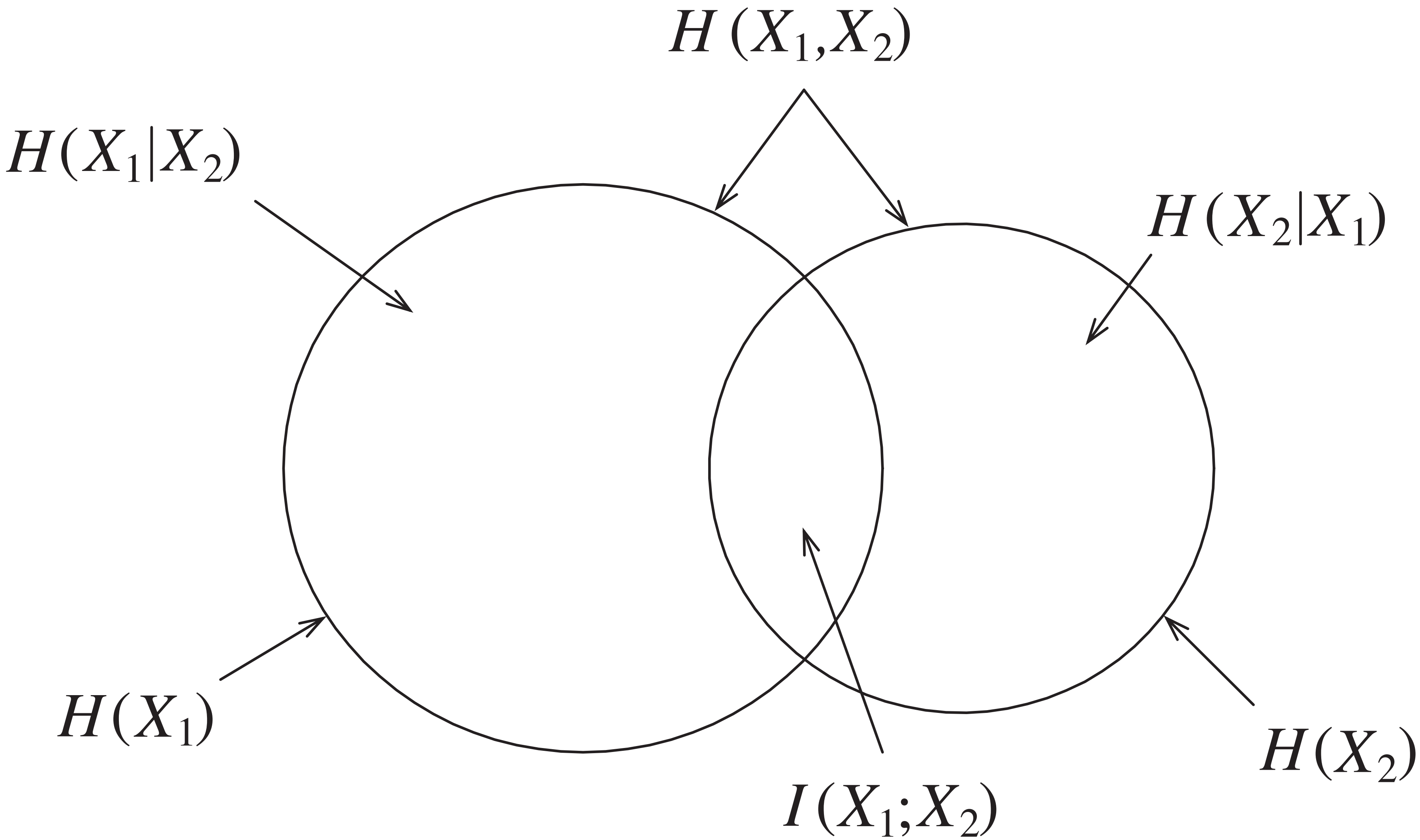
- These are called Shannon's information measures.

- Any Shannon's information measure is a linear combination of joint entropies.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- For example,

$$H(X_1, X_2) = H(X_1) + H(X_2) - I(X_1; X_2)$$

in information theory corresponds to

$$\mu(\tilde{X}_1 \cup \tilde{X}_2) = \mu(\tilde{X}_1) + \mu(\tilde{X}_2) - \mu(\tilde{X}_1 \cap \tilde{X}_2)$$

the Inclusion-Exclusion formulation in set theory, where

$$\begin{array}{ccc} H/I & \leftrightarrow & \mu \\ , & \leftrightarrow & \cup \\ ; & \leftrightarrow & \cap \\ | & \leftrightarrow & - \end{array}$$

μ is any set-additive function, and $\tilde{X}$ is a set variable corresponding to random variable X.

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2. What would
$$\mu(\Omega) = \mu(A) + \mu(A^c)$$
correspond to?

The I-Measure μ^*

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- To tackle

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need to fix n, the number of random variables, and let $\Omega = \bigcup_{i=1}^{n} \tilde{X}_i$.

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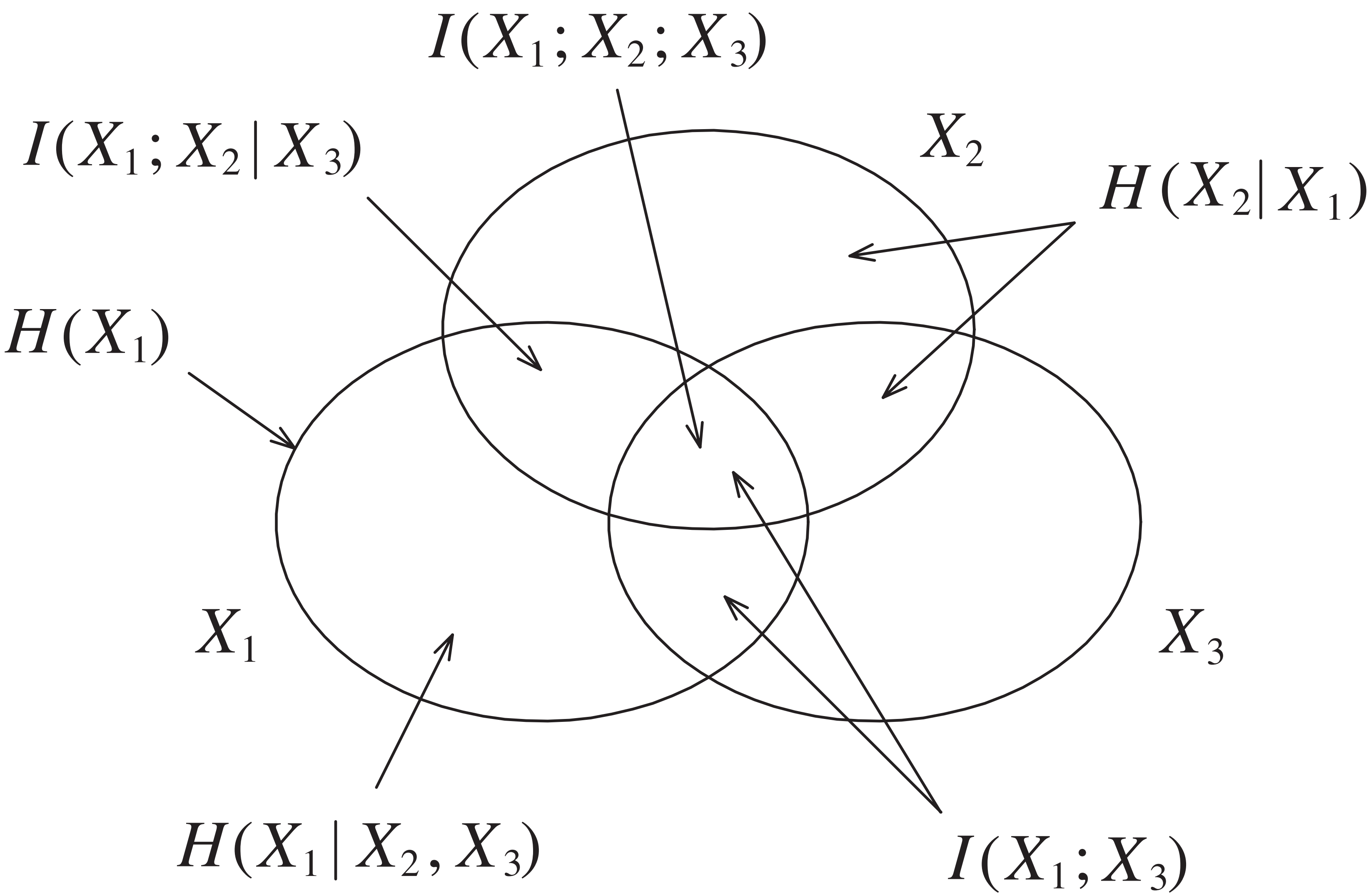
- As an example, for $n = 2$ and $A = \tilde{X}_1$,

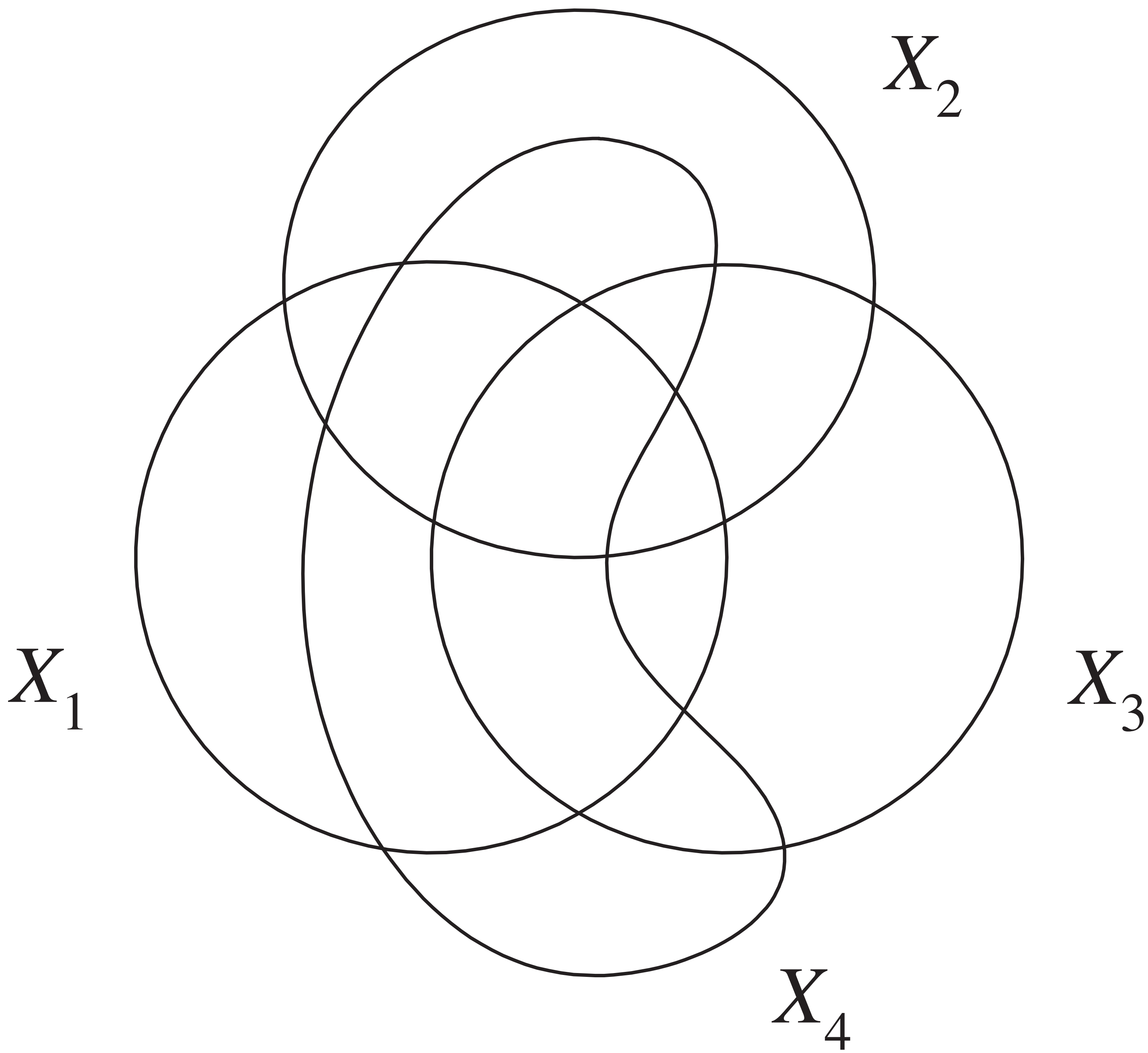
$$\mu(\Omega) = \mu(A) + \mu(A^c)$$

corresponds to

$$H(X_1, X_2) = H(X_1) + H(X_2|X_1).$$

Information Diagrams





X_2
X_1
X_3
X_4

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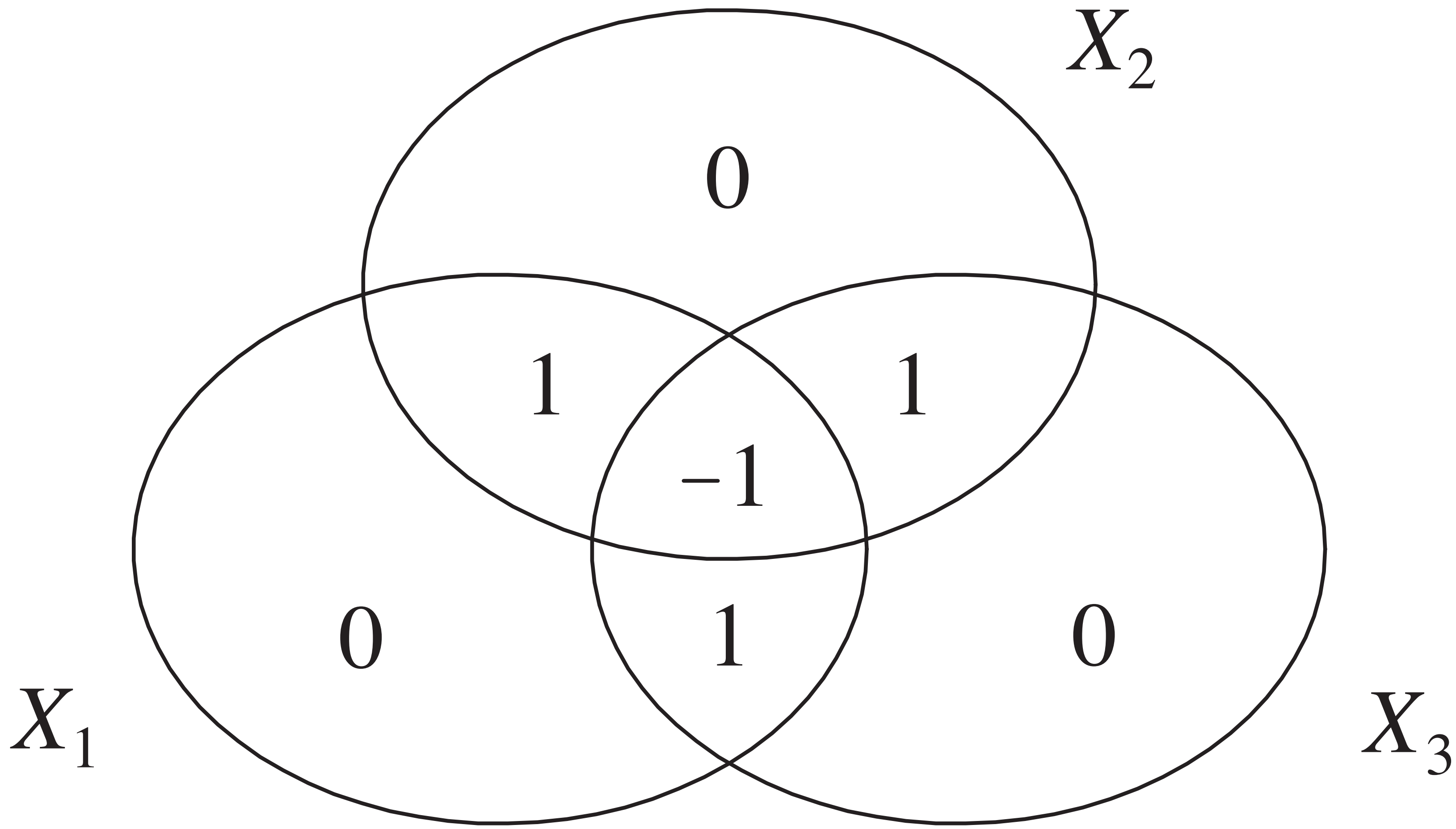
Example 3.10

- X_1, X_2 – i.i.d. binary r.v.'s uniform on $\{0, 1\}$
- $X_3 = X_1 + X_2 \bmod 2$
- Easy to check:
  - $H(X_i) = 1$, for all i
  - X_1, X_2, X_3 are pairwise independent, so that

$$H(X_i, X_j) = 2 \text{ and } I(X_i; X_j) = 0, \text{ for all } i \neq j$$

  - Under these constraints, $I(X_1; X_2; X_3) = -1$.



The information diagram for Example 3.10

Example 3.15 (Imperfect Secrecy Theorem) Let X be the plain text, Y be the cipher text, and Z be the key in a secret key cryptosystem. Since X can be recovered from Y and Z, we have

$$H(X|Y,Z) = 0.$$

Show that this constraint implies

$$I(X;Y) \geq H(X) - H(Z).$$

Remark Do not need to make these assumptions about the scheme:

- $H(Y|X,Z) = 0$
- $I(X;Z) = 0$

VII. Future Work

We now address some issues for further investigations.

a) We have constructed a real measure μ^* on $\mathcal{F}$, which we call the I-Measure, from the joint distribution of the random variables involved. It should be pointed out that not every real measure μ on $\mathcal{F}$ is an I-Measure. For μ to be an I-Measure, it is necessary that the value of μ on the elements of $\mathcal{F}$, which correspond to Shannon's information measures are nonnegative. However, given such a measure, it is not clear whether we can always find a joint distribution for the random variables such that Shannon's information measures on these random variables agree with the value of μ on the corresponding elements of $\mathcal{F}$. This is a very fundamental question to be answered.

b) The value of μ^* on the elements of $\mathcal{F}$ that correspond to Shannon's information measures are always nonnegative. A question of interest is: What are the elements of $\mathcal{F}$ on which the value of μ^* are always nonnegative? The more general question of what linear combinations of entropies are always nonnegative was raised by Te Sun Han [13].

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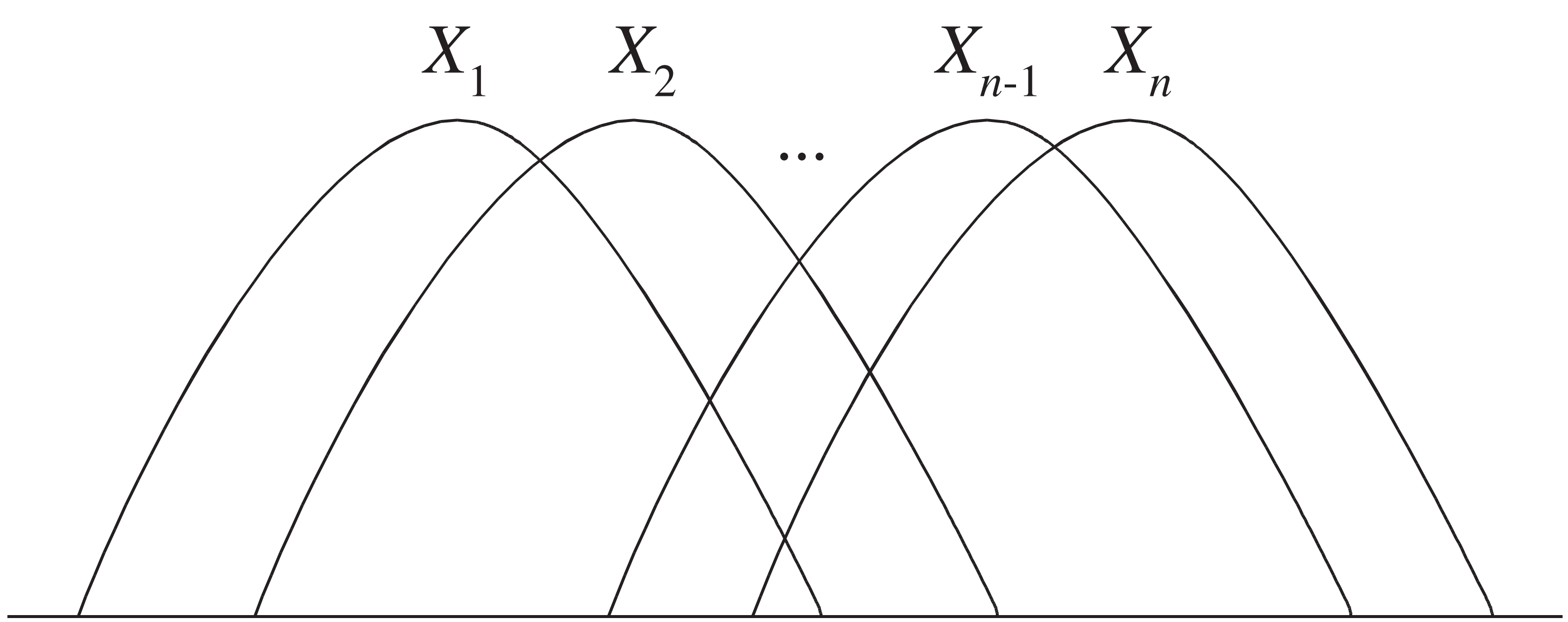

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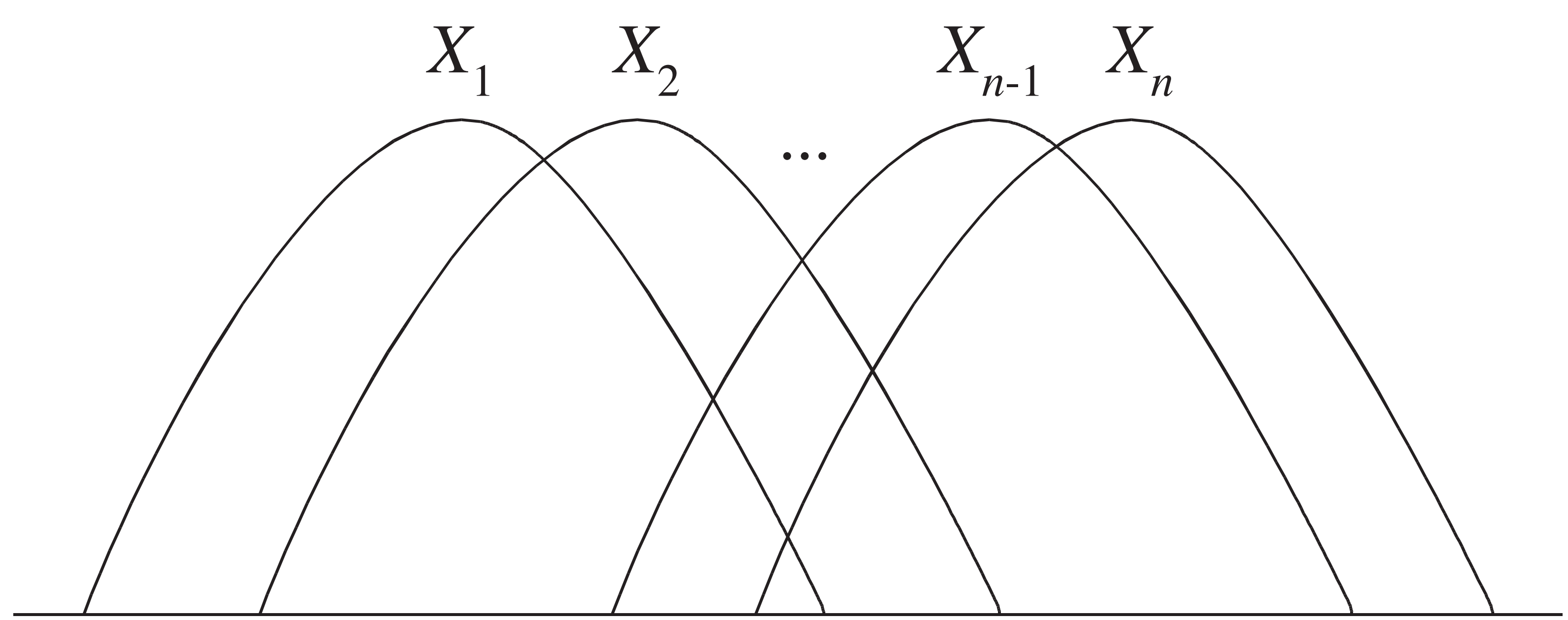
- This theme can be extended to Markov random field. (Y, Ye and Lee, 2002).



Gaussian Multiterminal Source Coding

（正規多重情報源符号化問題の研究）

February, 1980

Supervisor: Prof. Nobunori Oshima

Tsutomu Kawabata

Course of Mathematical Engineering
and Instrumentation Physics,
Division of Engineering,
Graduate School,
University of Tokyo

In his book [7], Abramson uses interesting figures to help their readers to memorize various relationship among amounts of information. For example the relationships among H(X), H(Y), H(Y|X), H(Y|X), H(X,Y) and I(X;Y) are precisely expressed in the quantitative relationships of areas [Fig. A.1]. The consistencies on additivities that each form of amount of information has an aspect as a measure. However the situation is not so simple when we consider three random variables X,Y and Z. Pictures in [Fig. A.2] which also from [7] exactly express the relationships such as

$$I(X;Y|Z)=H(X|Z)-H(X|Y,Z) \quad . \qquad (A.1)$$

However there appears in this cace a quantity $I(X;Y;Z)\overset{d}{=}H(X,Y,Z)-H(X,Y)-H(Y,Z)-H(Z,X)+H(X)+H(Y)+H(Z)$, whose meaning is not easy to understand. In fact I(X;Y;Z) is not always positive.

Our major interest is not in the generally correlated random

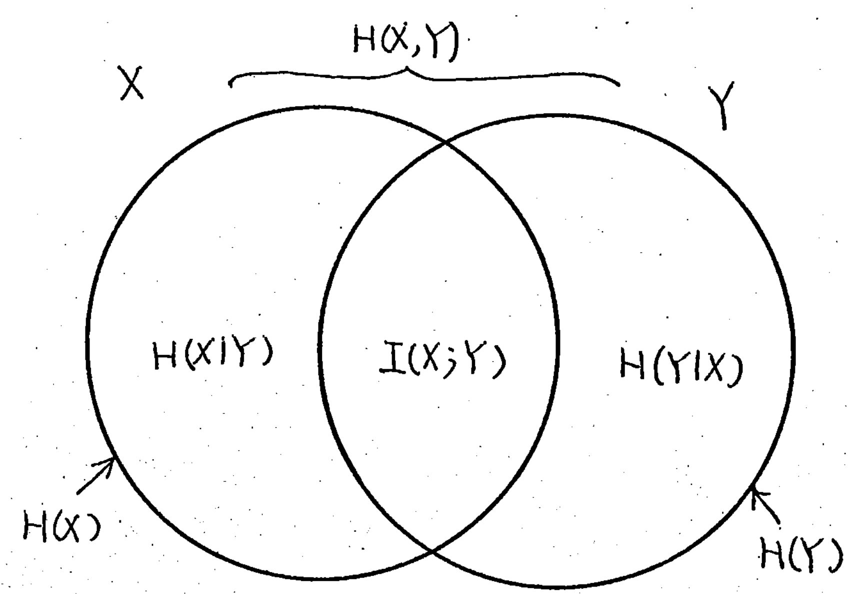


Fig. A.1

Relationships among information (Abramson [7])

We want to define a measure μ on the Boolean algebra generated by $\tilde{X}_1, \cdots \tilde{X}_m$, which we denote by $\mathcal{B}$, such that the equalities

$$H(X_A) = \mu(\tilde{X}_A) \tag{A.3}$$

$$H(X_A|X_B) = \mu(\tilde{X}_A - \tilde{X}_B) \tag{A.4}$$

$$I(X_A;X_B) = \mu(\tilde{X}_A \cap \tilde{X}_B) \tag{A.5}$$

$$I(X_A;X_B|X_C) = \mu(\tilde{X}_A \cap \tilde{X}_B - \tilde{X}_C) \tag{A.6}$$

are satisfied.

Next we define concepts and prove some lemmas.

Def. A.1.

Non-empty $A \in \mathcal{B}$ is called p-atom if it is not $\mathbb{R}^2 - \bigcup_{i=1}^{n} \tilde{X}_i$ and for any $B \in \mathcal{B}$, $B \supset A$ or $B \cap A = \phi$.

An p-atom is illustrated in [Fig. A.8]. It is noted that a p-atom is not a usual atom in the Boolean algebraic sence. It is really the stuff (not empty) set.

Lemma A.1.

If $B \in \mathcal{B}$ is not empty and not $\mathbb{R}^2 - \bigcup_{i=1}^{n} \tilde{X}_i$, it is expressed uniquely as a finite direct sum of p-atom.

[proof]

This is a Boolean algebraic result. By the condition (A.2) many atoms vanishes. [Fig. A.8] show the situation.

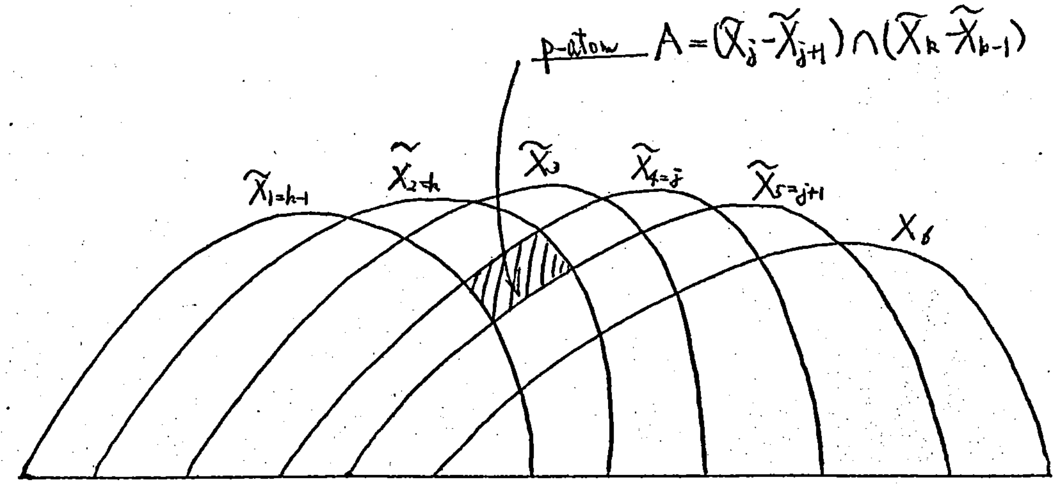


Fig. A.8

An p-atom

The Structure of the *I*-Measure of a Markov Chain

Tsutomu Kawabata and Raymond W. Yeung

***Abstract*—The underlying mathematical structure of Shannon's information measures was studied in a recent paper by Yeung, and the *I*-Measure μ^*, which is a signed measure defined on a proper σ-field $\mathcal{F}$, was introduced. The *I*-Measure is a natural extension of Shannon's information measures and is uniquely defined by them. They also introduced as a consequence the *I*-Diagram as a geometric tool to visualize the relationship among the information measures. In general, an *I*-Diagram for n random variables must be constructed in n-1 dimensions. It is shown that for any finite collection of random variables forming a Markov chain, μ^* assumes a very simple structure which can be illustrated by an *I*-Diagram in two dimensions, and μ^* is a nonnegative measure.**

***Index Terms*—Shannon's information measures, *I*-Measure, *I*-Diagram, Markov Chain.**

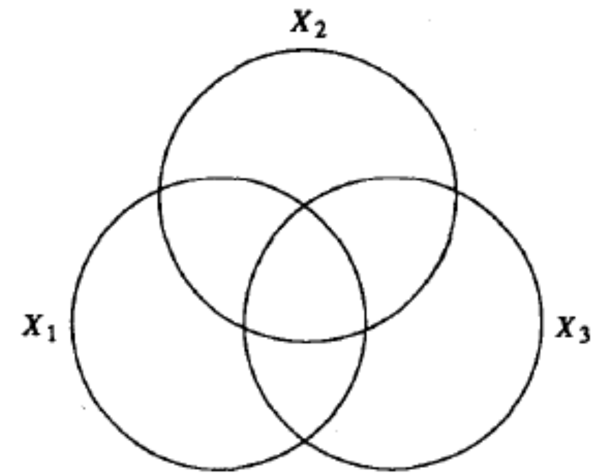


Fig. 1.

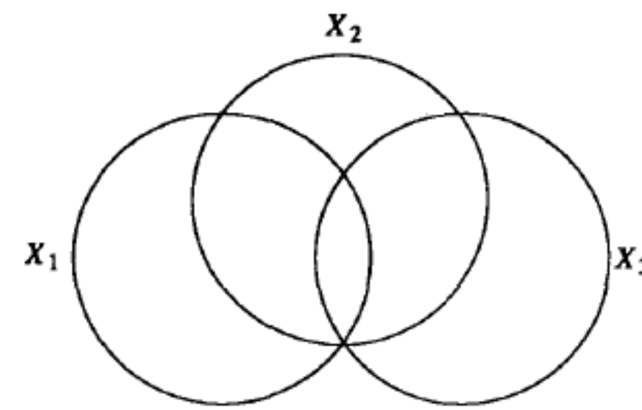


Fig. 2.

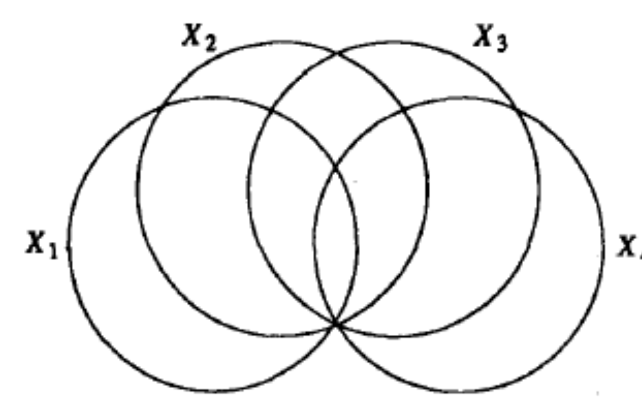


Fig. 3.

2) $I(X_2; X_3) = I(X_1; X_4) + I(X_1; X_3 \mid X_4) + I(X_2; X_4 \mid X_1) + I(X_2; X_3 \mid X_1, X_4)$,
3) $H(X_2, X_4 \mid X_1, X_3) = H(X_2 \mid X_1, X_3) + H(X_4 \mid X_3)$,
4) $H(X_2 \mid X_1, X_4) = H(X_2 \mid X_1, X_3) + I(X_2; X_3 \mid X_1, X_4)$.

Theorem 1: Let $X_i, i = 1, \cdots, n, n \geq 2$, be collection of random variables and $-(X_i)-$. Then $I(X_1; \cdots; X_n) = I(X_1; X_n)$.

Proof: We shall prove the theorem by induction. It is clear that it is true for $n = 2$. Assume it is true for some $n \geq 2$. Then

$$I(X_1; X_2; \cdots; X_{n+1}) = I(X_1; X_3; \cdots; X_{n+1}) - I(X_1; X_3; \cdots; X_{n+1} | X_2).$$

By the induction hypothesis,

$$I(X_1; X_3; \cdots; X_{n+1}) = I(X_1; X_{n+1}).$$

Now by Theorem 5 in [1],

$$I(X_1; X_3; \cdots; X_{n+1} | X_2) = \sum_{x_2} P[X_2 = x_2] I(X_1; X_3; \cdots; X_{n+1} | X_2 = x_2).$$

Conditioning on $X_2 = x_2$, the random variables $X_1, X_3, \cdots, X_{n+1}$ still form a Markov chain. By the induction hypothesis, we conclude that

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Acknowledgment

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1995

Zhen Zhang visited CUHK in Fall 1995

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- For n random variables, there are $2^n - 1$ joint entropies.
- E.g., $n = 3$, the $2^3 - 1 = 7$ joint entropies are

$$H(X_1), H(X_2), H(X_3), H(X_1, X_2), H(X_2, X_3),$$

$$H(X_1, X_3), H(X_1, X_2, X_3)$$

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Preliminaries

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- $H_\Omega : 2^{\mathcal{N}_n} \rightarrow \mathbb{R}$ is set function with $H_\Omega(\phi) = 0$.
- H_Ω is called the entropy function of Ω.

The Entropy Function as a Polymatroid

- It is well-known that for any Ω, H_Ω satisfies the following *polymatroidal axioms*. For any $\alpha, \beta \subset \mathcal{N}_n$,

  (P1) $H_\Omega(\phi) = 0$;

  (P2) $H_\Omega(\alpha) \leq H_\Omega(\beta)$ if $\alpha \subset \beta$;

  (P3) $H_\Omega(\alpha) + H_\Omega(\beta) \geq H_\Omega(\alpha \cap \beta) + H_\Omega(\alpha \cup \beta)$.

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- That is,

$$\begin{aligned} \text{entropy} &\geq 0 \\ \text{mutual info} &\geq 0 \\ \text{conditional entropy} &\geq 0 \\ \text{conditional mutual info} &\geq 0 \end{aligned}$$

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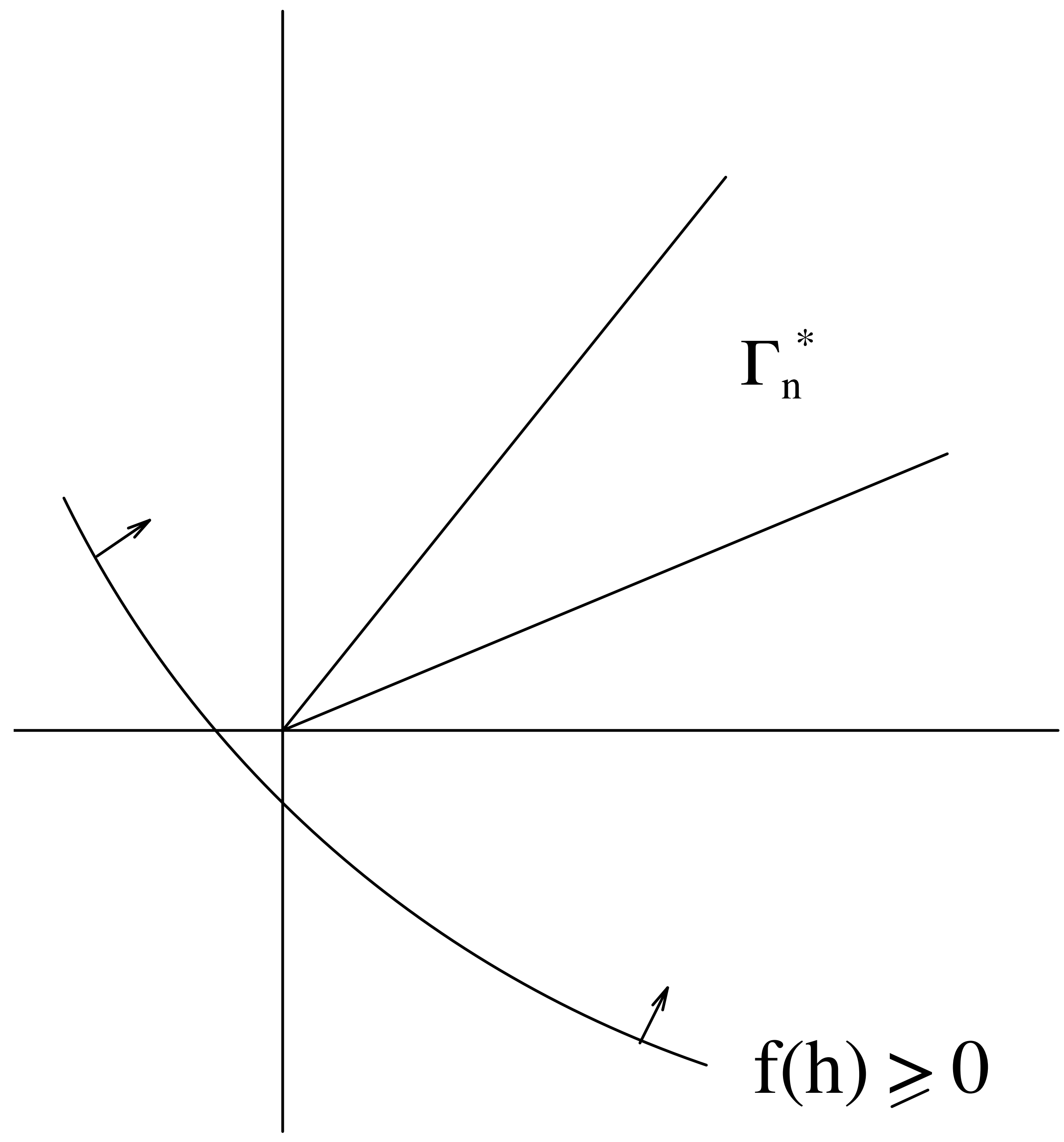
- In fact, $f(\mathbf{h}) \geq 0$ always holds if and only if

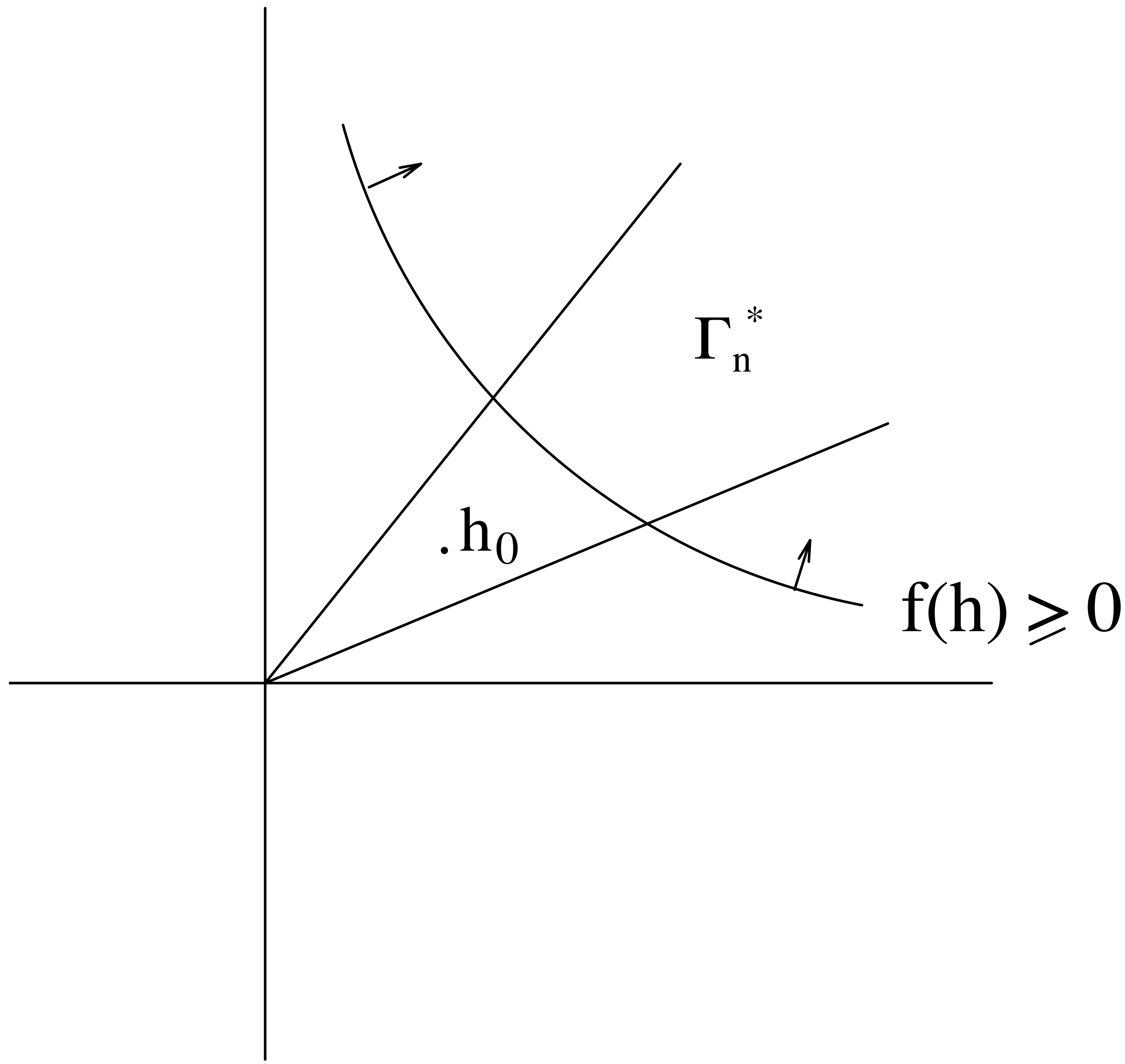
$$\overline{\Gamma}_n^* \subset \{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}$$

because $\{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}$ is closed.

$f(\mathbf{h}) \geq 0$ Always holds



$f(\mathbf{h}) \geq 0$ Does Not Always holds



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- An entropy inequality $f(\mathbf{h}) \geq 0$ is called a Shannon-type inequality if it is implied by the basic inequalities, or

$$\Gamma_n \subset \{\mathbf{h} \in \mathcal{H}_n : f(\mathbf{h}) \geq 0\}.$$

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ITIP and Xitip are linear programming based, while ITTP is axiom based.

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- Therefore, unconstrained non-Shannon-type inequalities can exist only for 4 or more random variables.

- In general,

  – Γ^*_n is neither closed nor convex, but $\overline{\Gamma}^*_n$ is a convex cone.

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- However, it is not strong enough to imply that $\overline{\Gamma}_4^* \neq \Gamma_4$.

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- This implies $\overline{\Gamma}_4^* \neq \Gamma_4$!

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- For any X_1, X_2, X_3, X_4, construct two auxiliary random variables $\tilde{X}_1, \tilde{X}_2$ by

$$(X_1, X_2) \xleftarrow{p(x_1,x_2|x_3,x_4)} (X_3, X_4) \xrightarrow{p(x_1,x_2|x_3,x_4)} (\tilde{X}_1, \tilde{X}_2)$$

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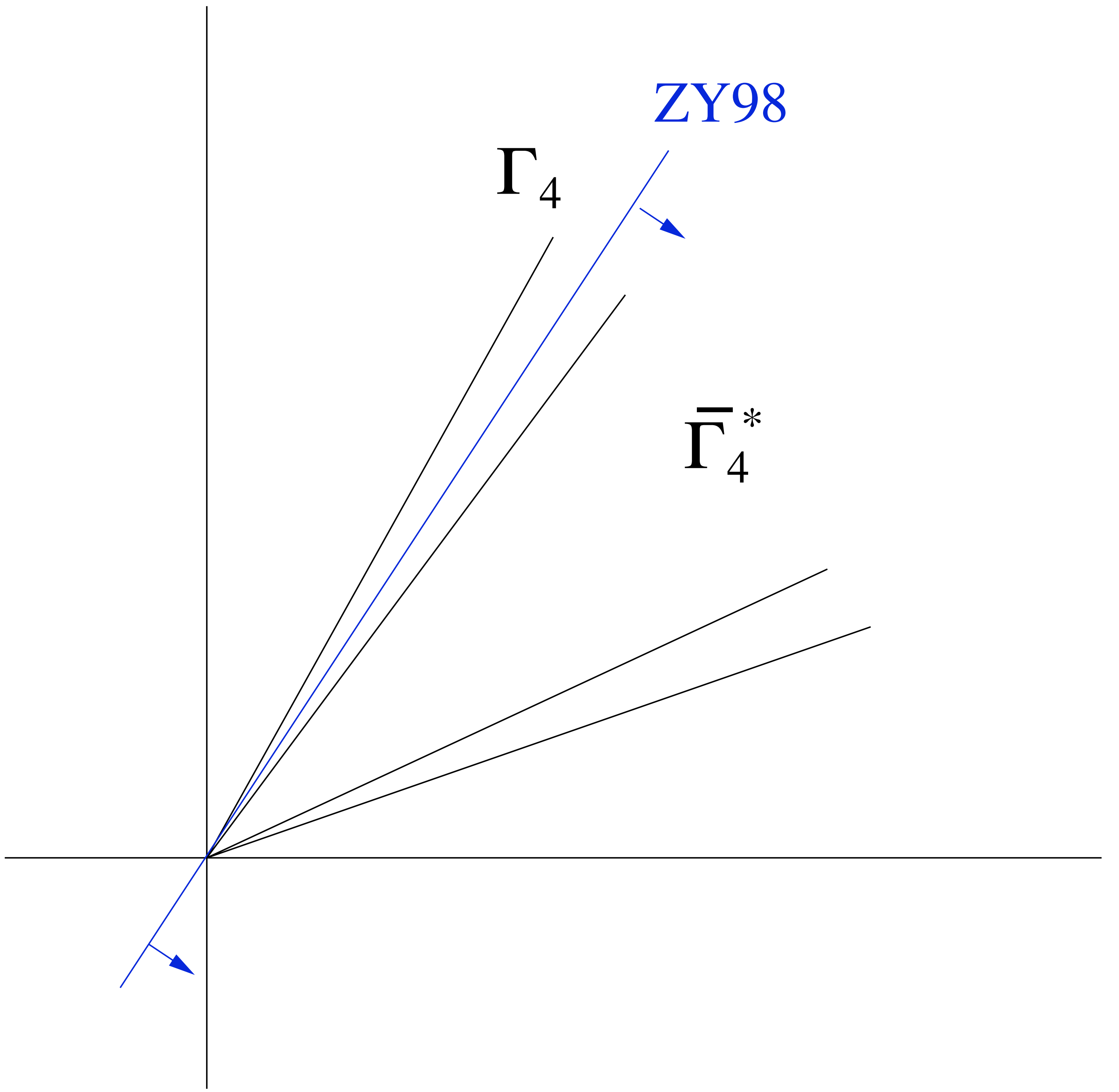
- Can be proved by using ITIP.

An Illustration of ZY98



ITIP

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1.
```
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True
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   ```

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   True
   ```

ITIP

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   `True`

2. `>> ITIP(’I(X;Z) = 0’,’I(X;Z|Y) = 0’,’I(X;Y) = 0’)`
   `True`

3. `>> ITIP(’I(Z;U) - I(Z;U|X) - I(Z;U|Y) <=`
   ` 0.5 I(X;Y) + 0.25 I(X;ZU) + 0.25 I(Y;ZU)’)`
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 0.5 I(X;Y) + 0.25 I(X;ZU) + 0.25 I(Y;ZU)')
Not provable by ITIP
```

ZY98

1986

From nicholas@cs.ubc.ca Tue Jul 7 23:04:11 1998
X400-Received: by /PRMD=ca/ADMD=telecom.canada/C=ca/; Relayed; Tue, 7 Jul
1998
8:03:59 UTC-0700
Date: Tue, 7 Jul 1998 8:03:59 UTC-0700
X400-Originator: nicholas@cs.ubc.ca
X400-Recipients: non-disclosure:;
X400-Content-Type: P2-1984 (2)
X400-MTS-Identifier: [/PRMD=ca/ADMD=telecom.canada/C=ca/;980707080359]
Content-Identifier: 4429
X-UIDL: 900031892.048
From: Nicholas Pippenger <nicholas@cs.ubc.ca>
To: zzhang@milly.usc.edu, whyeung@ie.cuhk.edu.hk
MIME-Version: 1.0 (Generated by Ean X.400 to MIME gateway)

I have just seen your paper "On the Characterization of Entropy Function via Information Inequalities" in the IEEE Transactions on Information Theory. Please allow me to congratulate you on a most beautiful result! I worked on the problem of whether \overbar{\Gamma}^*_n = \Gamma_n during the 80s, without any success. I presented it as an open problem at the SPOC (Specific Problems on Communication and Computation) Conference in 1986--I believe there were proceedings published by Springer, but they seem to be out of print now.

It was wonderful to see your paper.

 - Nick Pippenger

What Are the Laws of Information Theory?

Nicholas Pippenger
IBM Almaden Research Laboratory K51-801
650 Harry Road
San Jose, California 95120-6099

Shannon defined the *entropy* $H(X)$ of a random variable X assuming values in a finite set $\mathcal{X}$ to be $-\sum_{x\in\mathcal{X}} \Pr(X=x)\log\Pr(X=x)$. The entropy $H(X,Y,Z)$ of a finite set $\{X,Y,Z\}$ of random variables is defined by regarding the tuple (X,Y,Z) as a single random variable. In information theory, one also deals with *conditional entropies*, like $H(X \mid Y) = H(X,Y) - H(Y)$; *mutual informations*, like $I(X;Y) = H(X) + H(Y) - H(X,Y)$; and *conditional mutual informations*, like $I(X;Y \mid Z) = H(X,Y) + H(X,Z) - H(X,Y,Z) - H(Z)$. All identities and inequalities concerning these quantities, however, can be reduced to ones involving only "plain" entropies, like $H(X,Y,Z)$, by invoking these definitions. The identities are known (see [H] and [R]). The problem posed here is to determine the inequalities.

If $\{X_t\}_{t\in T}$ is a family of random variables, and if $S \subseteq T$, let H_S denote the entropy of the subfamily $\{X_s\}_{s\in S}$. The resulting map $H : 2^T \to \mathbf{R}$ satisfies the following conditions (known as the *polymatroid axioms*).

(1) $H_S \geq 0$ and $H_\emptyset = 0$.
(2) $H_R \leq H_S$ if $R \subseteq S$.
(3) $H_{R\cup S} + H_{R\cap S} \leq H_R + H_S$.

These conditions are immediate consequences of the fact that the logarithm vanishes at unity, is increasing and is concave. Are there any other conditions? If so, what are they? If not, show that any function satisfying (1), (2) and (3) can be approximated arbitrarily closely by the entropies of some family of random variables. (I say "approximated arbitrarily closely" to avoid the question of what happens on the boundary of the polytope defined by (1), (2) and (3).)

[H] K. T. Hu, "On the Amount of Information", *Theory of Prob. and Appl.*, 7 (1962) 439–447.

[R] F. M. Reza, *An Introduction to Information Theory*, McGraw-Hill, New York, 1961.

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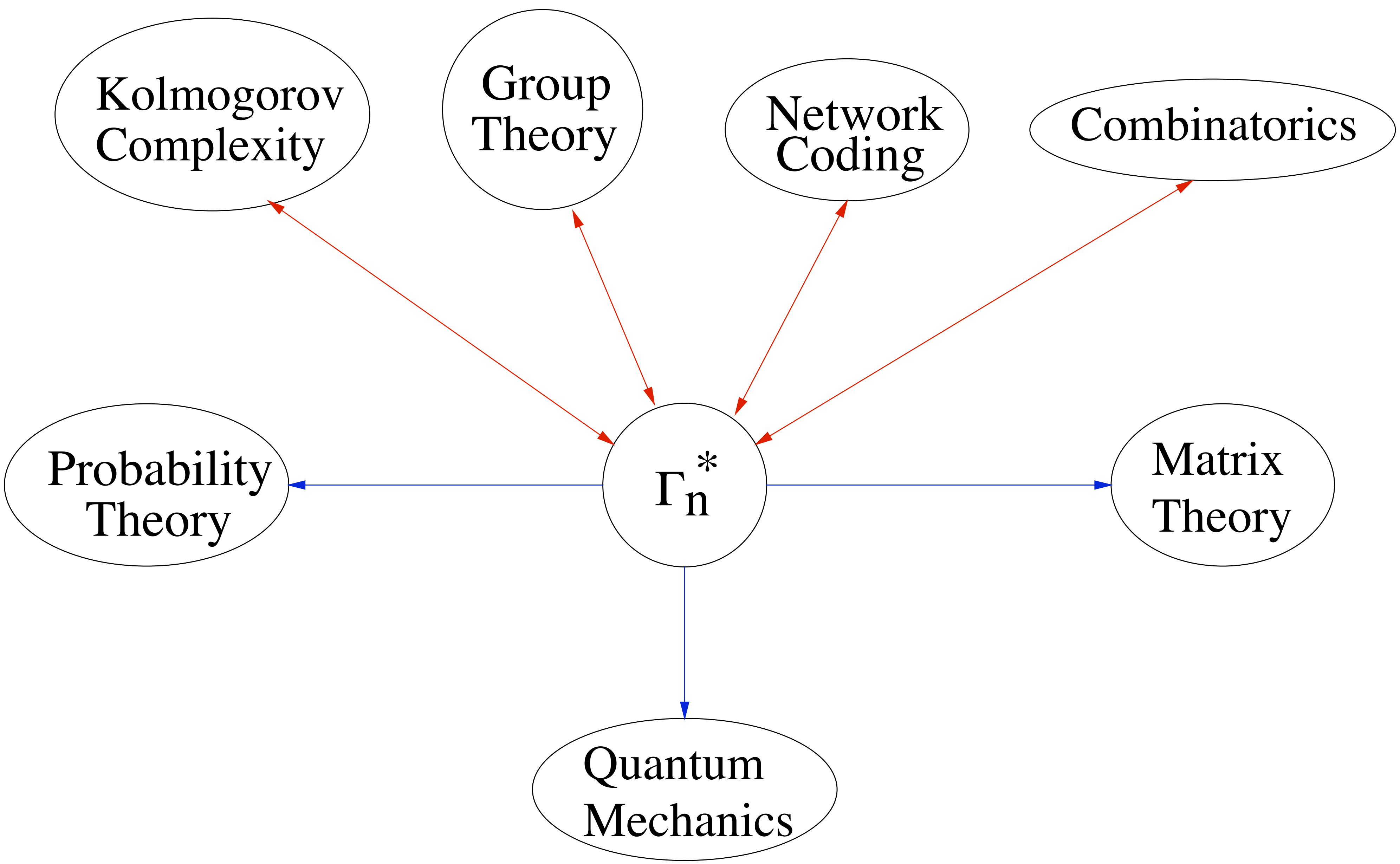
Remark

- Matúš (2007) also showed a fundamental property of Γ^*_n:

$$int(cl(\Gamma^*_n)) \subset \Gamma^*_n$$

i.e., Γ^*_n has a solid core.

Subjects Related to Γ_n^*



COMBINATORICS

2-D Quasi-Uniform Array

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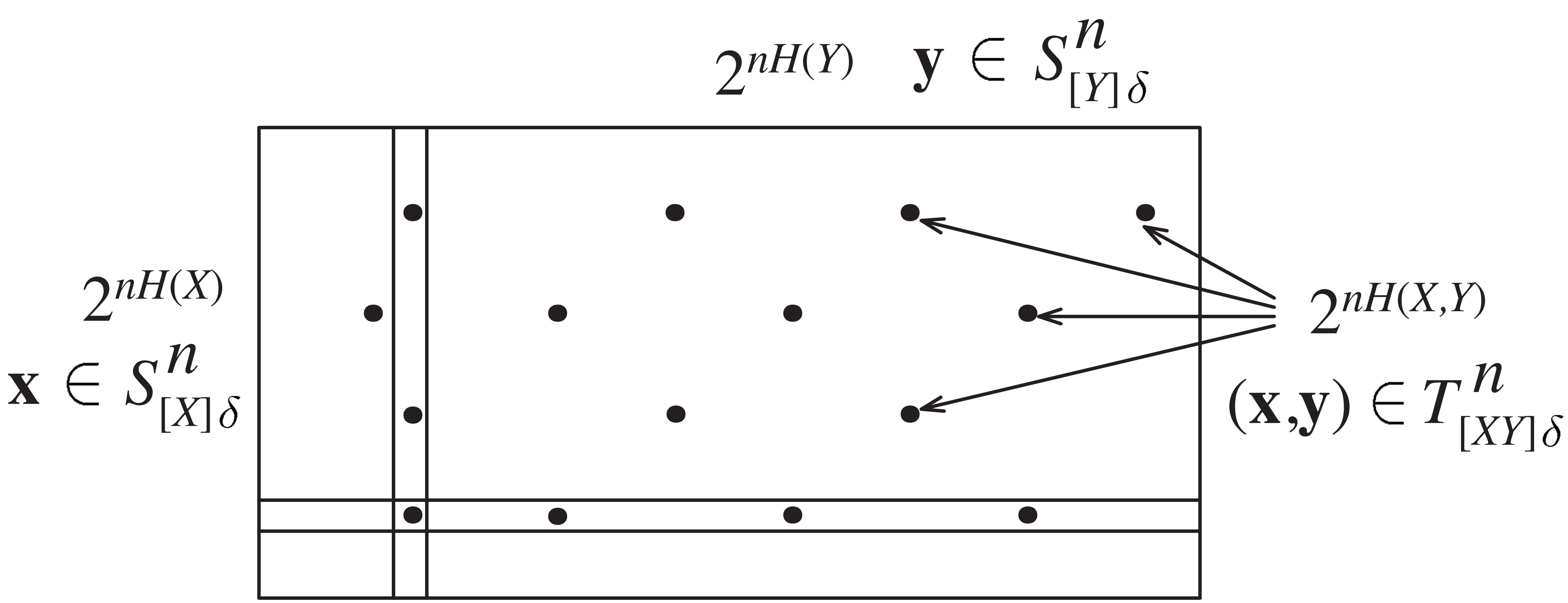
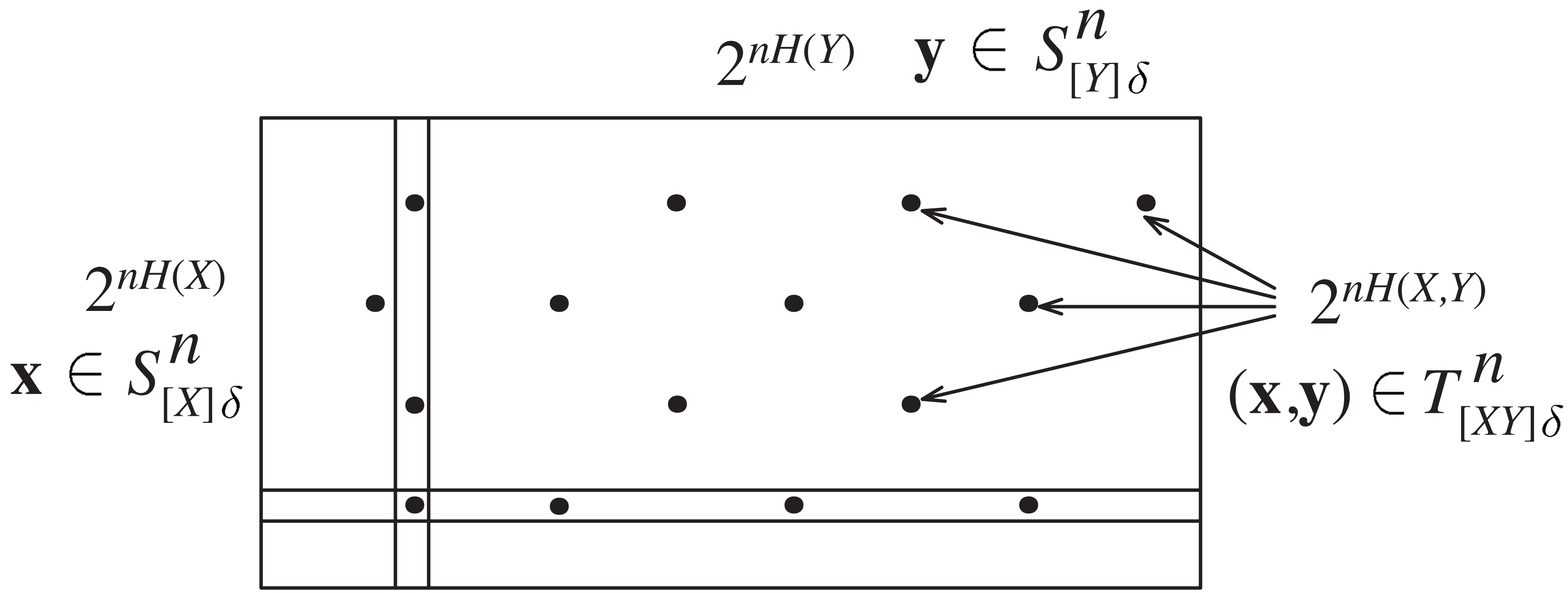
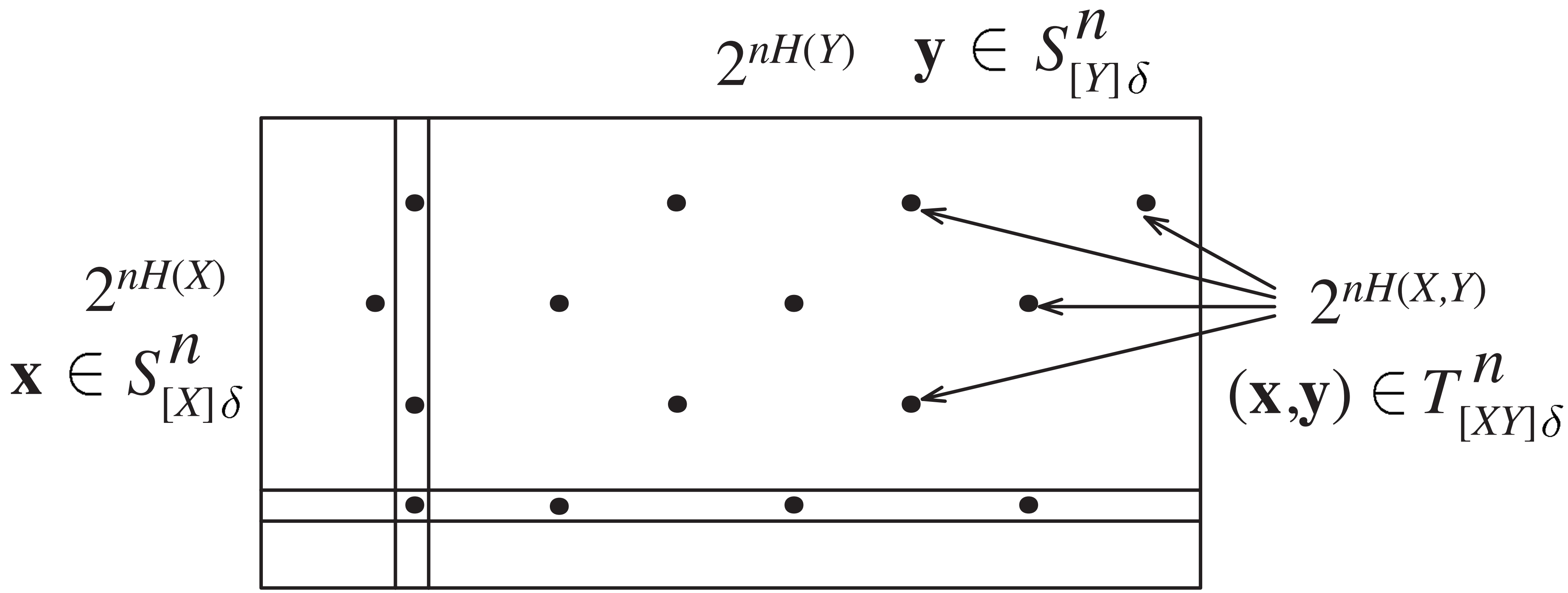




$2^{nH(Y)}$ $\mathbf{y} \in S^n_{[Y]\delta}$
$2^{nH(X)}$
$\mathbf{x} \in S^n_{[X]\delta}$
$2^{nH(X,Y)}$
$(\mathbf{x},\mathbf{y}) \in T^n_{[XY]\delta}$



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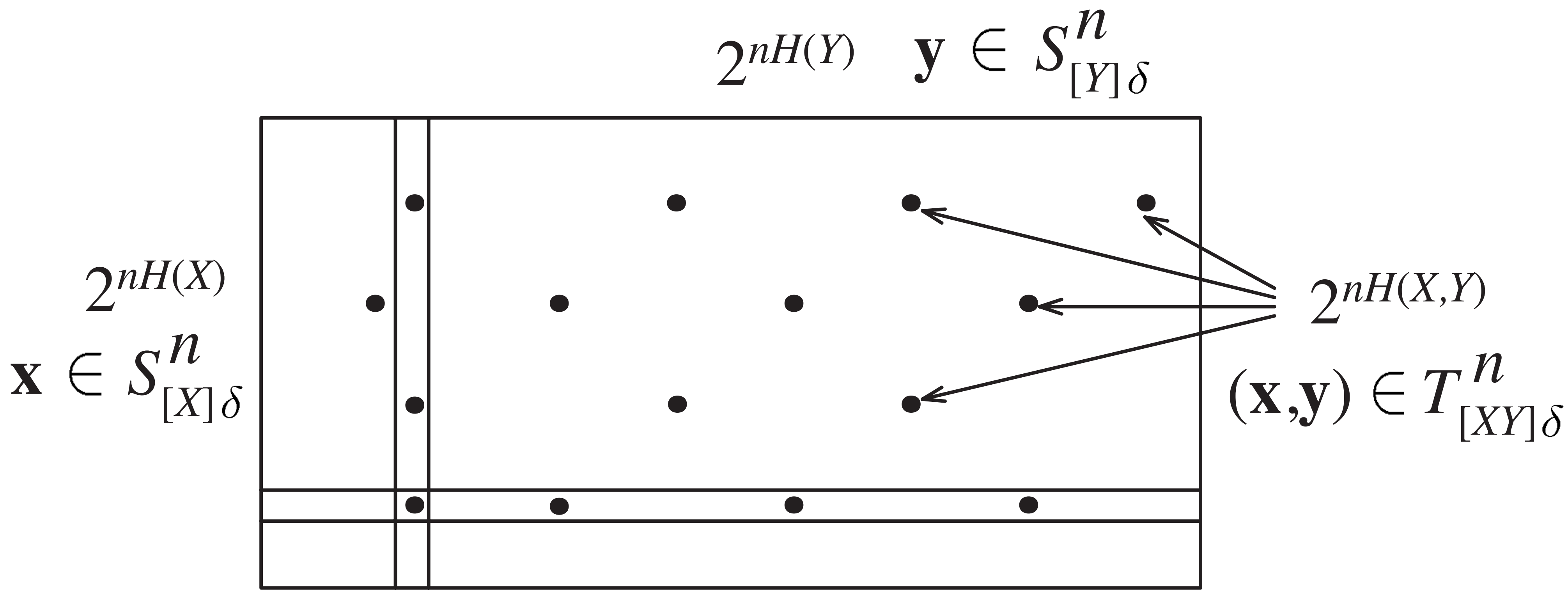

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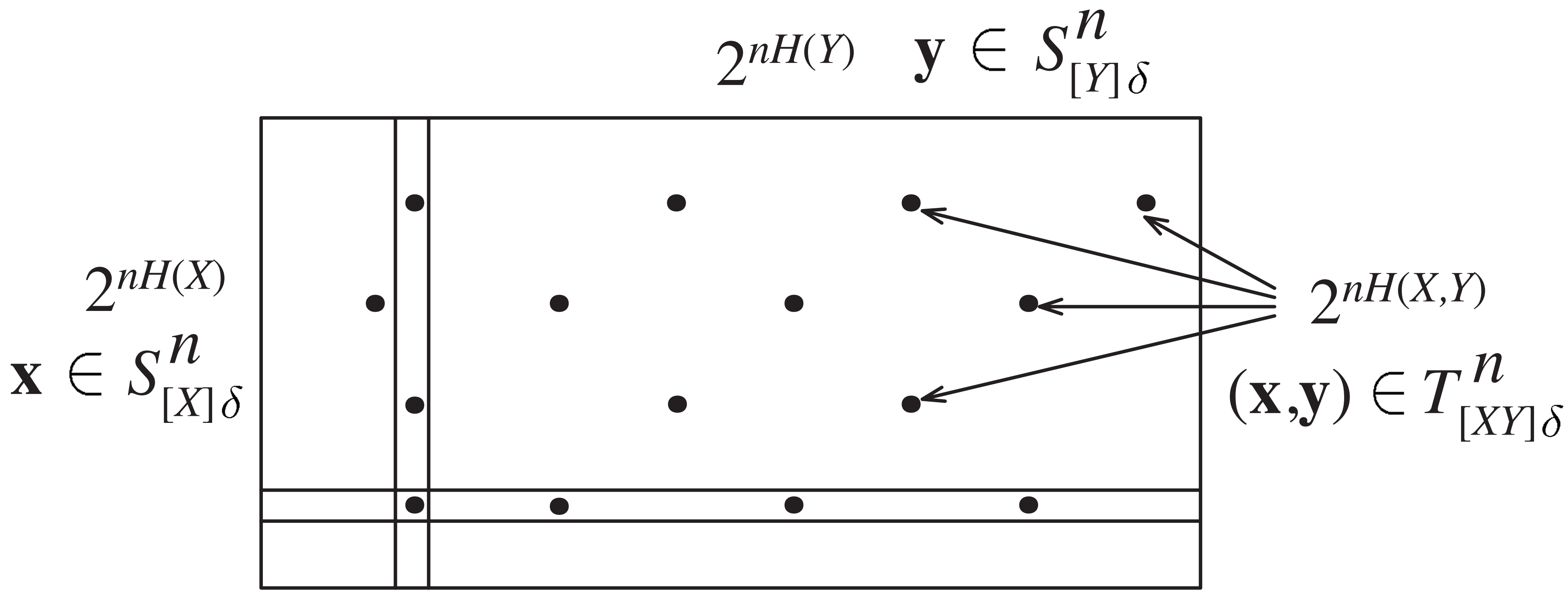

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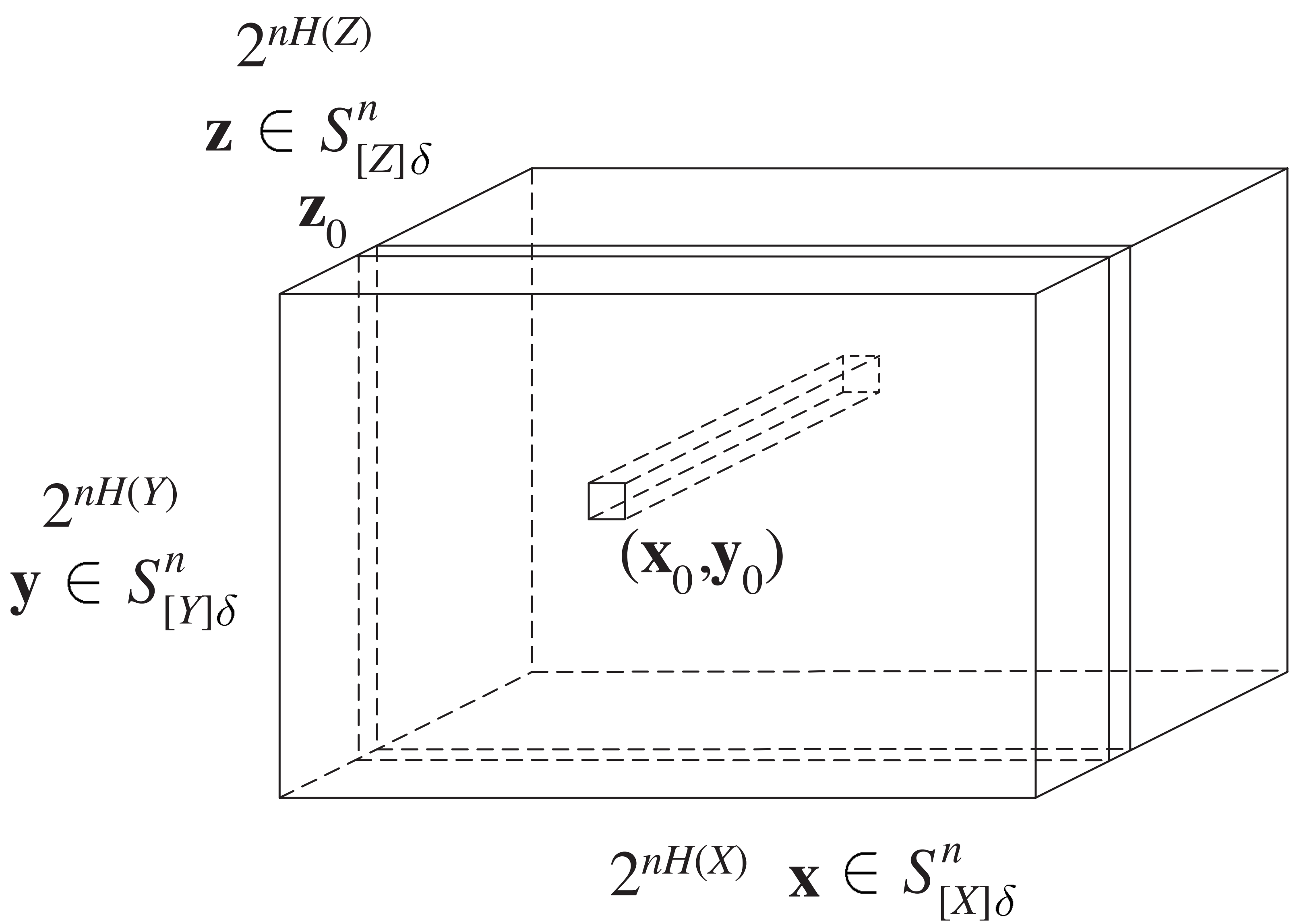
- Then the basic inequality $I(X;Y) \geq 0$ is about the unfilled entries in the array.

3-D Quasi-Uniform Array



Quasi-Uniform Arrays and Entropy Inequalities

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- Do quasi-uniform arrays fully capture all constraints on the entropy function?

- YES. T. Chan (2001) showed that all constraints on the entropy function can be obtained through quasi-uniform arrays, and vice versa.

GROUP THEORY

Entropy and Groups
(Chan-Y 2002)

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- Let G be a finite group and $G_1, G_2, \ldots, G_n$ be subgroups of G.

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- Let G be a finite group and $G_1, G_2, \ldots, G_n$ be subgroups of G.
- Let $G_\alpha = \cap_{i \in \alpha} G_i$, also a subgroup.
- A probability distribution for n random variables $X_1, X_2, \ldots, X_n$ can be constructed from any finite group G and subgroups $G_1, G_2, \ldots, G_n$, with

$$H(X_\alpha) = \log \frac{|G|}{|G_\alpha|}$$

  which depends only on the orders of G and $G_1, G_2, \ldots, G_n$.

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corresponds to for any finite group G and subgroups G_1, G_2,

$$\log \frac{|G|}{|G_1|} + \log \frac{|G|}{|G_2|} \geq \log \frac{|G|}{|G_1 \cap G_2|}$$

or

$$|G||G_1 \cap G_2| \geq |G_1||G_2|$$

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- This corresponds to

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- It can be proved that the correspondence between entropy inequalities and group inequalities is one-to-one.

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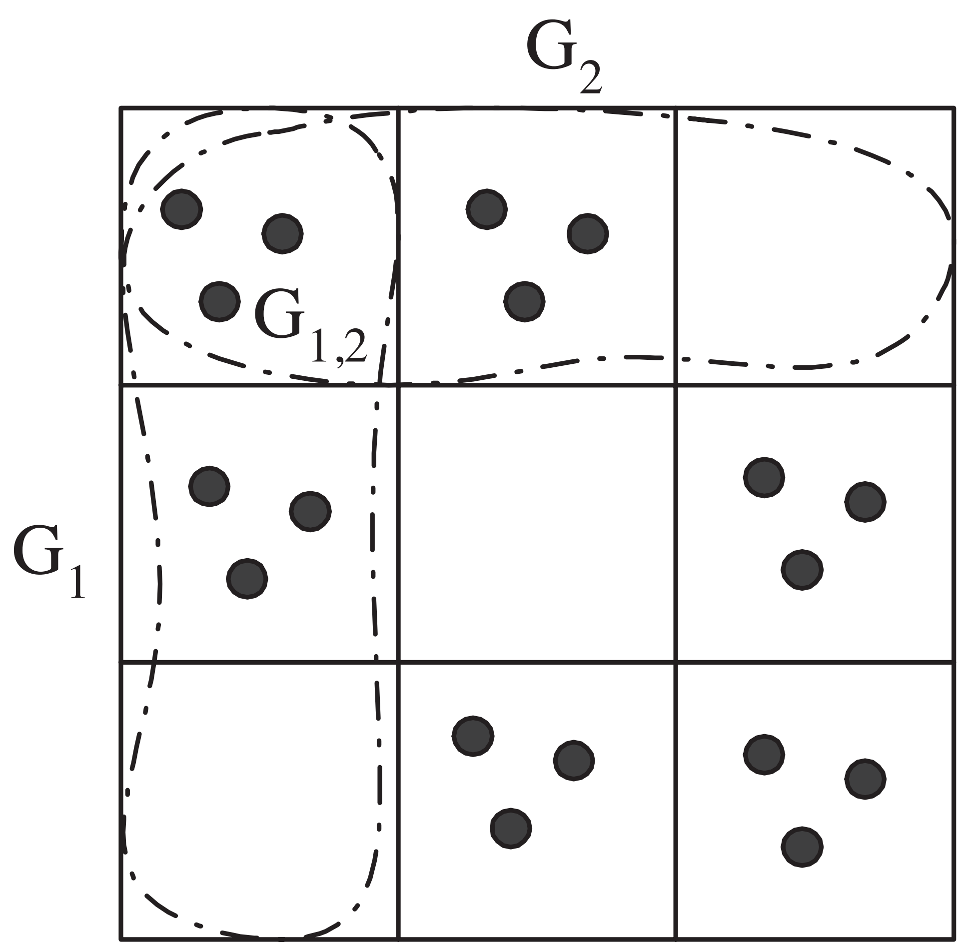

PROBABILITY THEORY

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- A very basic problem in probability theory.

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- Very hard for $n \geq 4$.

KOLMOGOROV COMPLEXITY

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- "Non-Shannon-type" Kolmogorov complexity inequalities can be obtained accordingly.

NETWORK CODING

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- For multi-source network coding, the network capacity can be characterized by Γ_n^*.
- The problem has been studied by Y and Zhang (1999), Song, Y, and Cai (2006). Yan, Y and Zhang (2007, 2012) finally obtained a complete characterization (implicit) of the network capacity in terms of Γ_n^*.

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- Thus

Every constraint on the entropy function is useful in
some multi-source network coding problems!

Secret Sharing

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- Recently, Beimel et al. (2008) has applied ZY98 to obtain a new performance bound in secret sharing.
- Secret sharing can be regarded as a special case of secure network coding (Cai and Y, 2002).

MATRIX THEORY

Differential Entropy Inequalities

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- Chan (2003) showed that a differential entropy inequality is valid iff the coefficients of the random variables are *balanced* and its discrete counterpart is valid.

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- The coefficients in ZY98 are balanced, so it is also valid for differential entropy.

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- Then

$$h(\mathbf{X}) = \frac{1}{2}\log\left[(2\pi e)^n |K|\right]$$

and for any subset α of $\{1, 2, \ldots, n\}$,

$$h(\mathbf{X}_\alpha) = \frac{1}{2}\log\left[(2\pi e)^{|\alpha|} |K_\alpha|\right]$$

where K_α is the corresponding submatrix of K.

Non-Shannon-Type Matrix Inequalities

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- Substituting these joint differential entropies into the inequality

$$h(X_1, X_2, \ldots, X_n) \leq \sum_i h(X_i)$$

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- Substituting these joint differential entropies into ZY98 gives

$$\begin{aligned} & |K_1||K_{12}||K_3|^2|K_4|^2|K_{134}|^4|K_{234}| \\ \leq \quad & |K_{13}|^3|K_{14}|^3|K_{34}|^3|K_{23}||K_{24}| \end{aligned}$$

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- Many other “non-Shannon-type” inequalities of the principal minors of positive definite matrices can be obtained this way.

Principal Minor Inequalities

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- Chan, Guo, and Y (2012) showed that this is the case for 3×3 positive definite matrices.
- Previously, Hassibi and Shadbakht (2008) studied normalized Gaussian entropy functions and obtained a characterization for 3 Gaussian random variables.

QUANTUM MECHANICS

The von Neumann Entropy

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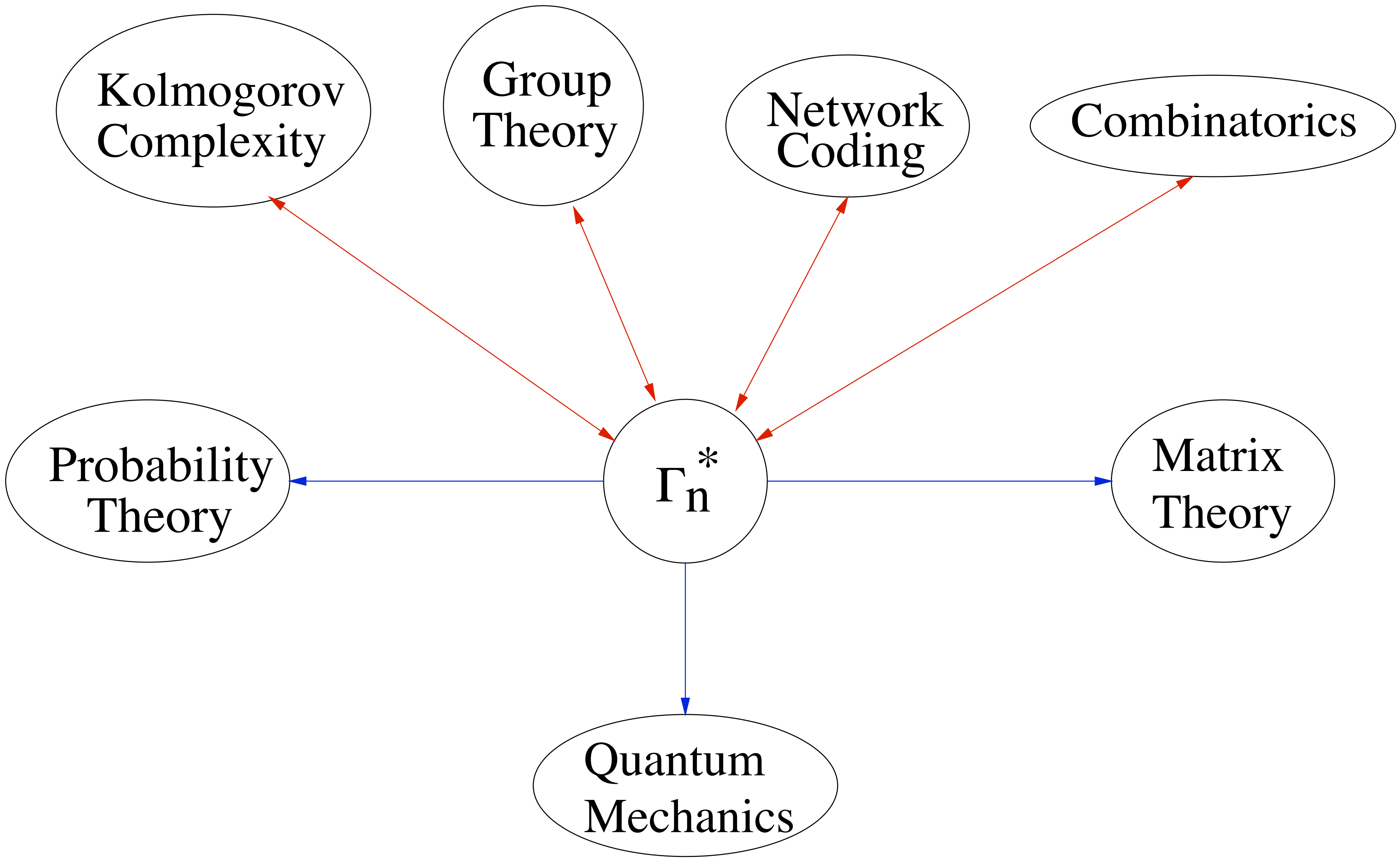
- Linden and Winter (2005) discovered for a 4-party system a constrained inequality for the von Neumann entropy which is independent of strong subadditivity.

Summary


Kolmogorov Complexity
Group Theory
Network Coding
Combinatorics
Probability Theory
Γ_n^*
Matrix Theory
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- "Non-Shannon-type" inequalities in different fields need further understanding.
- Information theory is clearly an integral part of mathematics.

Can we get out of this mess?

Facets of Entropy

Raymond W. Yeung*

October 4, 2012

Constraints on the entropy function are of fundamental importance in information theory. For a long time, the polymatroidal axioms, or equivalently the nonnegativity of the Shannon information measures, are the only known constraints. Inequalities that are implied by nonnegativity of the Shannon information measures are categorically referred to as Shannon-type inequalities. If the number of random variables is fixed, a Shannon-type inequality can in principle be verified by a software package known as ITIP. A non-Shannon-type inequality is a constraint on the entropy function which is not implied by the nonnegativity of the Shannon information measures. In the late 1990s, the discovery of a few such inequalities revealed that Shannon-type inequalities alone do not constitute a complete set of constraints on the entropy function. In the past decade or so, connections between the entropy function and a number of subjects in information sciences, mathematics, and physics have been established. These subjects include probability theory, network coding, combinatorics, group theory, Kolmogorov complexity, matrix theory, and quantum mechanics. This expository work is an attempt to present a picture for the many facets of the entropy function.[1]

Keywords: Entropy, polymatroid, non-Shannon-type inequalities, positive definite matrix, quasi-uniform array, Kolmogorov complexity, conditional independence, network coding, quantum information theory.

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[1]This work is based on the author's plenary talk with the same title at the 2009 IEEE International Symposium on Information Theory, Seoul, Korea, Jun 28 - Jul 3, 2009.