Alphabet Size Reduction for Secure Network Coding:

A Graph Theoretic Approach

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Outline

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Wiretap Network

- Let $G = (V, E)$ be a finite directed acyclic network with a single source node s and a set of sink nodes $T \subset V \setminus \{s\}$, where
  - V is the set of nodes, and
  - E is the set of edges.
- Parallel edges between two adjacent nodes are allowed.
- An index taken from an alphabet can be transmitted on each edge in E.

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  - E is the set of edges.
- Parallel edges between two adjacent nodes are allowed.
- An index taken from an alphabet can be transmitted on each edge in E.
- Let $\mathscr{A}$ be a collection of subsets of E, where every edge set in $\mathscr{A}$ is called a wiretap set.

Wiretap Network

A wiretap network is a quadruple $(G, s, T, \mathscr{A})$, where

- s generates a random source message M according to an arbitrary distribution on a message set $\mathcal{M}$;
- each $t \in T$ is required to recover the source message M with zero error;
- arbitrary one wiretap set in $\mathscr{A}$, but no more than one, may be fully accessed by a wiretapper;
- $\mathscr{A}$ is known by s and all $t \in T$ but which wiretap set in $\mathscr{A}$ is actually eavesdropped is unknown.

Wiretap Network

- It is necessary to randomize the source message to combat the wiretapper.
- The random key K available at the source node is a random variable that takes values in a set of keys $\mathcal{K}$ according to the uniform distribution.

Secure Network Codes

- Let $\mathcal{F}$ be an alphabet.
- An $\mathcal{F}$-valued secure network code on a wiretap network $(G, s, T, \mathscr{A})$ consists of a set of local encoding mappings $\{\phi_e : e \in E\}$ such that
  - if $e \in \mathrm{Out}(s)$,

$$\phi_e : \ \mathcal{M} \times \mathcal{K} \to \mathcal{F};$$

  - otherwise, i.e., if $e \in \mathrm{Out}(v)$ for a node $v \in V \setminus \{s\}$,

$$\phi_e : \ \mathcal{F}^{|\mathrm{In}(v)|} \to \mathcal{F}.$$

Secure Network Codes

Definition 1

For a secure network code on the wiretap network $(G, s, T, \mathscr{A})$, $I(Y_A; M) = 0$ *for every wiretap set* $A \in \mathscr{A}$, *where* $I(Y_A; M)$ *denotes the mutual information between* $Y_A = (Y_e : e \in A)$ *and* M.

The Required Alphabet Size

Proposition 1 ([Cai & Yeung][1])

Let $(G, s, T, \mathscr{A})$ *be a wiretap network and* $\mathcal{F}$ *be an alphabet with* $|\mathcal{F}| \geq |T|$*, the number of sink nodes in* G*. Then there exists an* $\mathcal{F}$*-valued secure network code over* $(G, s, T, \mathscr{A})$ *provided that* $|\mathcal{F}| > |\mathscr{A}|$*.*

[1] N. Cai and R. W. Yeung, "Secure Network Coding on a Wiretap Network," *IEEE Trans. Inf. Theory*, 2011.

The Required Alphabet Size

- The lower bound $|\mathscr{A}|$ on the required alphabet size is typically too large for implementation in terms of computational complexity and storage requirement.
- Reduction of the required alphabet size is a problem not only of theoretical interest but also of practical importance.

An Assumption

Assume that all wiretap sets are regular.

- A wiretap set A is said to be regular, if $|A| = \mathrm{mincut}(s, A)$.
- The collection of wiretap sets $\mathscr{A}$ is said to be regular, if all wiretap sets in $\mathscr{A}$ are regular.

An Assumption

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- The collection of wiretap sets $\mathscr{A}$ is said to be regular, if all wiretap sets in $\mathscr{A}$ are regular.
- Replace non-regular wiretap sets in $\mathscr{A}$ by their minimum cuts (that are regular) to form $\mathscr{A}'$.
- A secure network code that is secure for $\mathscr{A}'$ is also secure for $\mathscr{A}$.

Outline

Equivalence Relation "$\sim$"

- Let $(G, s, T, \mathscr{A})$ be a wiretap network.
- The binary relation "$\sim$":

  For any two edge sets A and A' in G, we write $A \sim A'$ provided that

  - there exists an edge set CUT that is a minimum cut between s and A and also between s and A'.

Equivalence Relation "$\sim$"

Proposition 2 ([Guang et al.][2])

The binary relation "$\sim$" is an equivalence relation. To be specific, For any three edge sets A, A', and A'' in G:

1. **(Reflexivity)** $A \sim A$;
2. **(Symmetry)** *if* $A \sim A'$ *then* $A' \sim A$;
3. **(Transitivity)** *if* $A \sim A'$ *and* $A' \sim A''$, $A \sim A''$.

[2] X. Guang, J. Lu, and F.-W. Fu, "Small field size for secure network coding", *IEEE Commun. Lett.*, 2015.

Equivalence Relation "$\sim$"

Proposition 3

Let $A_1, A_2, \cdots, A_m$ *be* m *equivalent edge sets under the equivalence relation "*$\sim$*". Then*

$$\text{mincut}(s, \cup_{i=1}^{m} A_i) = \text{mincut}(s, A_j), \quad \forall j,\ 1 \le j \le m.$$

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$$\mathrm{mincut}(s, \cup_{i=1}^{m} A_i) = \mathrm{mincut}(s, A_j), \quad \forall j, \ 1 \leq j \leq m.$$

- With "$\sim$", the wiretap sets in $\mathscr{A}$ can be partitioned into equivalence classes.
- All the wiretap sets in an equivalence class have a common minimum cut.

The Required Alphabet Size

Denote $N(\mathscr{A})$ by the number of the equivalence classes in $\mathscr{A}$.

Theorem 4

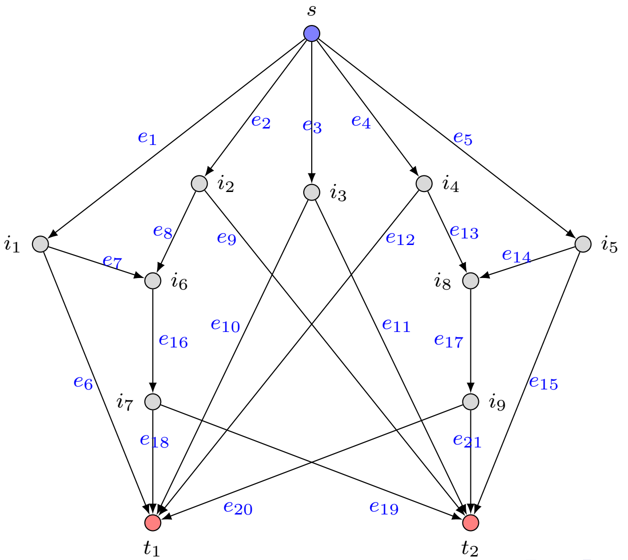
Let $(G, s, T, \mathscr{A})$ be a wiretap network and $\mathcal{F}$ be an alphabet with $|\mathcal{F}| \geq |T|$. Then there exists an $\mathcal{F}$-valued secure network code over $(G, s, T, \mathscr{A})$ provided that

$$|\mathcal{F}| > N(\mathscr{A}).$$

- This lower bound $N(\mathscr{A})$ was originally obtained in [Guang *et al.*][3] for r-wiretap networks, but it also applies for general wiretap networks.

[3] X. Guang, J. Lu, and F.-W. Fu, "Small field size for secure network coding", *IEEE Commun. Lett.*, 2015.

Example



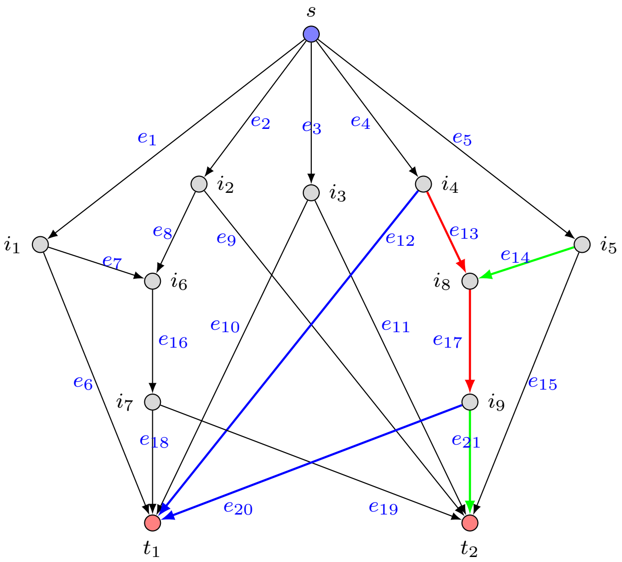
s
e1
e2
e3
e4
e5
i2
i3
i4
i1
i5
e8
e9
e12
e13
e7
e14
i6
i8
e10
e11
e16
e17
e6
e15
i7
i9
e18
e21
e20
e19
t1
t2

- Let the collection of wiretap sets $\mathscr{A}$ be:

$$
\begin{aligned}
\mathscr{A} = \Big\{ & \{e_6\}, \{e_7\}, \{e_8\}, \{e_9\}, \{e_{12}\}, \{e_{13}\}, \\
& \{e_{14}\}, \{e_{15}\}, \{e_{18}\}, \{e_{19}\}, \{e_{20}\}, \{e_{21}\}, \\
& \{e_6, e_{18}\}, \{e_6, e_{19}\}, \{e_7, e_{18}\}, \{e_7, e_{19}\}, \{e_8, e_{11}\}, \\
& \{e_8, e_{16}\}, \{e_8, e_{18}\}, \{e_9, e_{10}\}, \{e_9, e_{18}\}, \{e_9, e_{19}\}, \\
& \{e_{10}, e_{14}\}, \{e_{10}, e_{15}\}, \{e_{10}, e_{19}\}, \{e_{10}, e_{21}\}, \{e_{11}, e_{14}\}, \\
& \{e_{11}, e_{15}\}, \{e_{11}, e_{18}\}, \{e_{11}, e_{20}\}, \{e_{12}, e_{20}\}, \{e_{12}, e_{21}\}, \\
& \{e_{13}, e_{17}\}, \{e_{13}, e_{21}\}, \{e_{14}, e_{20}\}, \{e_{14}, e_{21}\}, \{e_{15}, e_{20}\}, \\
& \{e_{15}, e_{21}\}, \{e_{18}, e_{20}\}, \{e_{18}, e_{21}\}, \{e_{19}, e_{20}\}, \{e_{19}, e_{21}\}, \\
& \{e_1, e_3, e_{16}\}, \{e_1, e_{11}, e_{16}\}, \{e_2, e_{10}, e_{16}\}, \\
& \{e_3, e_5, e_{17}\}, \{e_4, e_{10}, e_{17}\}, \{e_5, e_{11}, e_{17}\} \Big\}.
\end{aligned}
$$

- $|\mathscr{A}| = 48$.

e.g., consider three wiretap sets $\{e_{12}, e_{20}\}$, $\{e_{13}, e_{17}\}$, $\{e_{14}, e_{21}\}$.



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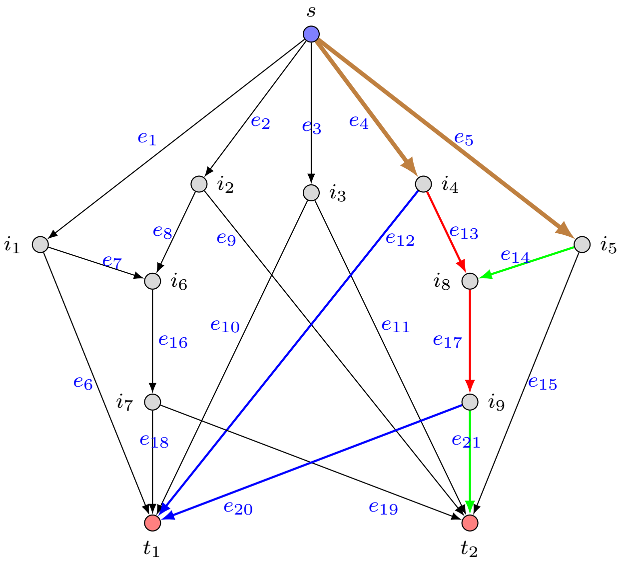


Example

- The equivalence classes of wiretap sets are:

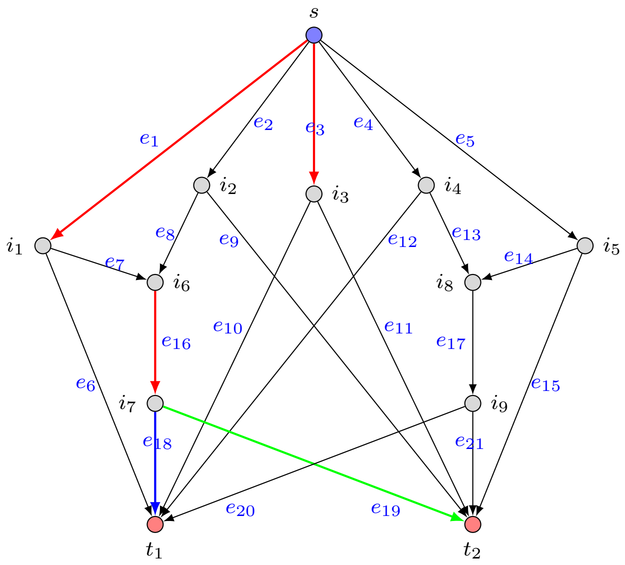
$$\begin{aligned}
&\mathrm{Cl}_1 = \Big\{\{e_6\},\{e_7\}\Big\}, \quad \mathrm{Cl}_2 = \Big\{\{e_8\},\{e_9\}\Big\},\\
&\mathrm{Cl}_3 = \Big\{\{e_{12}\},\{e_{13}\}\Big\}, \ \mathrm{Cl}_4 = \Big\{\{e_{14}\},\{e_{15}\}\Big\},\\
&\mathrm{Cl}_5 = \Big\{\{e_{18}\},\{e_{19}\}\Big\}, \ \mathrm{Cl}_6 = \Big\{\{e_{20}\},\{e_{21}\}\Big\},\\
&\mathrm{Cl}_7 = \Big\{\{e_8,e_{11}\},\{e_9,e_{10}\}\Big\},\\
&\mathrm{Cl}_8 = \Big\{\{e_{10},e_{19}\},\{e_{11},e_{18}\}\Big\},\\
&\mathrm{Cl}_9 = \Big\{\{e_{10},e_{21}\},\{e_{11},e_{20}\}\Big\},\\
&\mathrm{Cl}_{10} = \Big\{\{e_{10},e_{14}\},\{e_{10},e_{15}\},\{e_{11},e_{14}\},\{e_{11},e_{15}\}\Big\},\\
&\mathrm{Cl}_{11} = \Big\{\{e_{18},e_{20}\},\{e_{18},e_{21}\},\{e_{19},e_{20}\},\{e_{19},e_{21}\}\Big\};
\end{aligned}$$

Example

$$\begin{aligned}
\mathrm{Cl}_{12} =& \Big\{ \{e_6, e_{18}\}, \{e_6, e_{19}\}, \{e_7, e_{18}\}, \{e_7, e_{19}\}, \\
& \{e_8, e_{16}\}, \{e_8, e_{18}\}, \{e_9, e_{18}\}, \{e_9, e_{19}\} \Big\}, \\
\mathrm{Cl}_{13} =& \Big\{ \{e_{12}, e_{20}\}, \{e_{12}, e_{21}\}, \{e_{13}, e_{17}\}, \{e_{13}, e_{21}\}, \\
& \{e_{14}, e_{20}\}, \{e_{14}, e_{21}\}, \{e_{15}, e_{20}\}, \{e_{15}, e_{21}\} \Big\}, \\
\mathrm{Cl}_{14} =& \Big\{ \{e_1, e_3, e_{16}\}, \{e_1, e_{11}, e_{16}\}, \{e_2, e_{10}, e_{16}\} \Big\}, \\
\mathrm{Cl}_{15} =& \Big\{ \{e_3, e_5, e_{17}\}, \{e_4, e_{10}, e_{17}\}, \{e_5, e_{11}, e_{17}\} \Big\}.
\end{aligned}$$

- Then $N(\mathscr{A}) = 15$ $(< |\mathscr{A}| = 48)$.

Furthermore, consider $\mathrm{Cl}_5 = \{\{e_{18}\}, \{e_{19}\}\}$ and $\{e_1, e_3, e_{16}\}$.



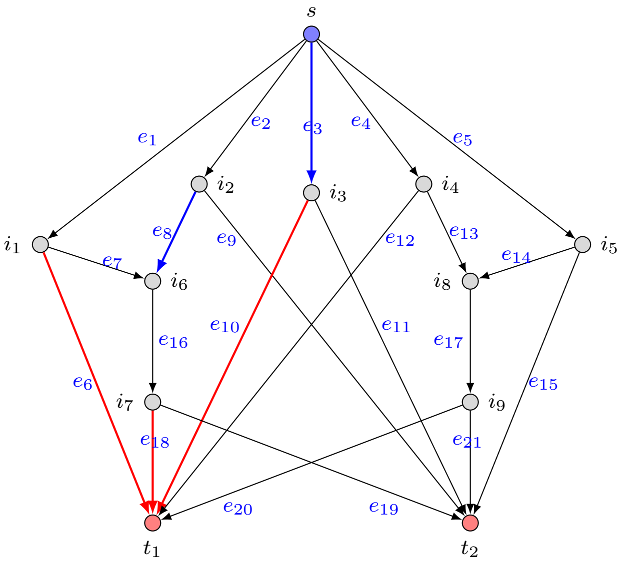
Wiretap-Set Domination

Definition 2 (Wiretap-Set Domination)

Let A_1 and A_2 be two wiretap sets in $\mathscr{A}$ with $|A_1| < |A_2|$.
We say that A_1 is dominated by A_2, denoted by $A_1 \prec A_2$, if there exists a minimum cut between s and A_2 that also separates A_1 from s. In other words, upon deleting the edges in the minimum cut between s and A_2, s and A_1 are also disconnected.

- Note that $A_1 \prec A_2$ does not mean that A_2 is at the "upstream" of A_1.

Let $A_1 = \{e_3, e_8\}$ and $A_2 = \{e_6, e_{10}, e_{18}\}$, and $A_1 \prec A_2$.



Equivalence-Class Domination

Definition 3 (Equivalence-Class Domination)

For two distinct equivalence classes Cl_1 and Cl_2, if there exists a common minimum cut of the wiretap sets in Cl_2 that separates all the wiretap sets in Cl_1 from s, we say that Cl_1 is dominated by Cl_2, denoted by $\mathrm{Cl}_1 \prec \mathrm{Cl}_2$.

Equivalence-Class Domination

Theorem 5

$\mathrm{Cl}(A_1) \prec \mathrm{Cl}(A_2)$ *if and only if* $A_1 \prec A_2$.

Equivalence-Class Domination

Theorem 6

The equivalence-class domination relation "$\prec$" amongst the equivalence classes in $\mathscr{A}$ is a strict partial order. Specifically, let Cl_1, Cl_2, and Cl_3 be three arbitrary equivalence classes, and then

1. **(Irreflexivity)** $\mathrm{Cl}_1 \nprec \mathrm{Cl}_1$;
2. **(Transitivity)** *if* $\mathrm{Cl}_1 \prec \mathrm{Cl}_2$ *and* $\mathrm{Cl}_2 \prec \mathrm{Cl}_3$, *then* $\mathrm{Cl}_1 \prec \mathrm{Cl}_3$;
3. **(Asymmetry)** *if* $\mathrm{Cl}_1 \prec \mathrm{Cl}_2$, *then* $\mathrm{Cl}_2 \nprec \mathrm{Cl}_1$.

Maximal Equivalence Class

- Now, the set of all the equivalence classes in $\mathscr{A}$ can be considered as a strictly partially ordered set.
- Thus, we can define its maximal equivalence classes.

Definition 4 (Maximal Equivalence Class)

For a collection of wiretap sets $\mathscr{A}$, an equivalence class Cl is a maximal equivalence class if there exists no other equivalence class Cl' such that $\mathrm{Cl}' \succ \mathrm{Cl}$.
Denote by $N_{\max}(\mathscr{A})$ the number of the maximal equivalence classes with respect to $\mathscr{A}$.

The Required Alphabet Size

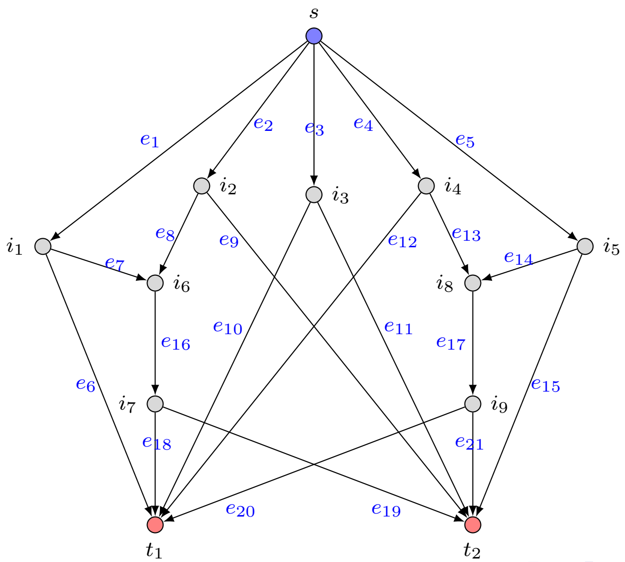
Theorem 7

Let $(G, s, T, \mathscr{A})$ *be a wiretap network and* $\mathcal{F}$ *be an alphabet with* $|\mathcal{F}| \geq |T|$*. Then there exists an* $\mathcal{F}$*-valued secure network code on* $(G, s, T, \mathscr{A})$ *provided that the alphabet size*

$$|\mathcal{F}| > N_{\max}(\mathscr{A}).$$

- $N_{\max}(\mathscr{A}) \leq N(\mathscr{A}) \leq |\mathscr{A}|$.

Example (Cont.)



Example (Cont.)

The Hasse diagram of all 15 equivalence classes, ordered by the equivalence-class domination relation "$\prec$".

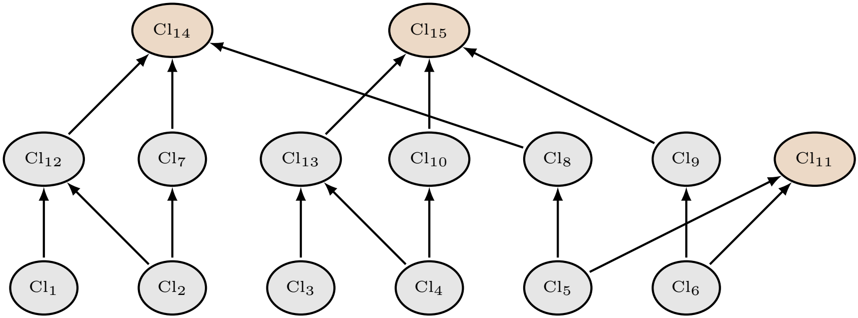


Figure: Cl_{11}, Cl_{14}, and Cl_{15} are all of the maximal equivalence classes.

Example (Cont.)

The alphabet size $\|\mathcal{F}\|$	
Lower Bound I: $\|\mathscr{A}\|$	48
Lower Bound II: $N(\mathscr{A})$	15
Lower Bound III: $N_{\max}(\mathscr{A})$	3

- The improvement of $N_{\max}(\mathscr{A})$ over $N(\mathscr{A})$ can be unbounded.

New Problem Proposed

- $N_{\max}(\mathscr{A})$ is graph-theoretical.
- $N_{\max}(\mathscr{A})$ only depends on the topology of the network G and the collection $\mathscr{A}$ of wiretap sets.

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- In general, computing the value of $N_{\max}(\mathscr{A})$, or characterizing the corresponding Hasse diagram, is nontrivial.
- Even in the simple example, its value is not obvious.
- **How to efficiently compute $N_{\max}(\mathscr{A})$?**

Outline

Primary Minimum Cut

Definition 5 (Primary Minimum Cut)

A minimum cut between the source node s and a sink node t in G is primary, if it separates s and all the minimum cuts between s and t.
In other words, a primary minimum cut between s and t is a common minimum cut of all the minimum cuts between s and t.

- The notion of primary minimum cut is crucial to the development of our algorithm.

Existence and Uniqueness of Primary Minimum Cut

Theorem 8

The primary minimum cut is well-defined, that is, the primary minimum cut between the source node s and a sink node t exists and is unique.

- The concept of the primary minimum cut between the source node s and a sink node t can be extended to between s and a wiretap set $A \in \mathscr{A}$.

Cornerstone

Theorem 9

In a wiretap network $(G, s, T, \mathscr{A})$, *let* Cl *be an arbitrary equivalence class of the wiretap sets. Then*

1. *all the wiretap sets in* Cl *have the same primary minimum cut, which hence is called the primary minimum cut of the equivalence class* Cl, *and*
2. *for every equivalence class* Cl' *with* $\mathrm{Cl}' \prec \mathrm{Cl}$, *the primary minimum cut of* Cl *separates all the wiretap sets in* Cl' *from* s.

Cornerstone

- To compute $N_{\max}(\mathscr{A})$, it suffices to compute the primary minimum cuts of all the maximal equivalence classes.
- With this, we bypass the complicated operations for determining the equivalence classes of wiretap sets and the domination relation among them.
- This is the key to the efficiency of the algorithm.

Algorithm for computing $N_{\max}(\mathscr{A})$:

1. Define a set $\mathscr{B}$, and initialize $\mathscr{B}$ to the empty set.
2. Arbitrarily choose a wiretap set $A \in \mathscr{A}$ that has the largest cardinality in $\mathscr{A}$. Find the primary minimum cut between s and A, and call it CUT.
3. Partition the edge set E into two disjoint subsets: E_{CUT} and $E^c_{\mathrm{CUT}} \triangleq E \setminus E_{\mathrm{CUT}}$, where E_{CUT} is the set of the edges reachable from the source node s upon deleting the edges in CUT.
4. Remove all the wiretap sets in $\mathscr{A}$ that are subsets of E^c_{CUT} and add the primary minimum cut CUT to $\mathscr{B}$.
5. Repeat Steps 2) to 4) until $\mathscr{A}$ is empty and output $\mathscr{B}$, where $N_{\max}(\mathscr{A}) = |\mathscr{B}|$.

Algorithm

Algorithm for computing $N(\mathscr{A})$:

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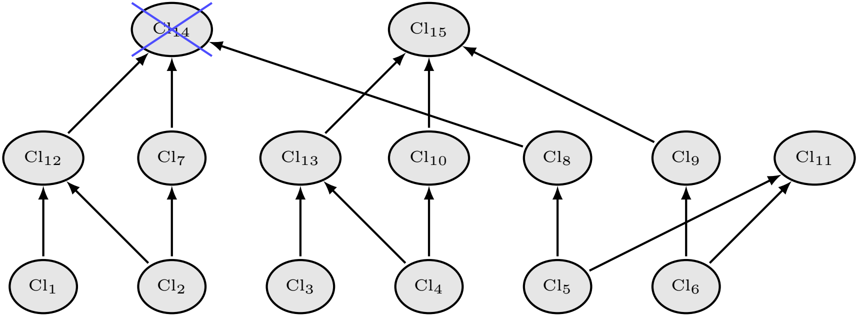
3. Partition the edge set E into two disjoint subsets: E_{CUT} and $E_{\mathrm{CUT}}^c \triangleq E \setminus E_{\mathrm{CUT}}$, where E_{CUT} is the set of the edges reachable from the source node s upon deleting the edges in CUT.
4. Remove all the wiretap sets of the same cardinality as A in $\mathscr{A}$ that are subsets of E_{CUT}^c. Add the primary minimum cut CUT to $\mathscr{B}$.
5. Repeat Steps 2) to 4) until $\mathscr{A}$ is empty and output $\mathscr{B}$, where $N(\mathscr{A}) = |\mathscr{B}|$.

Example (Cont.)



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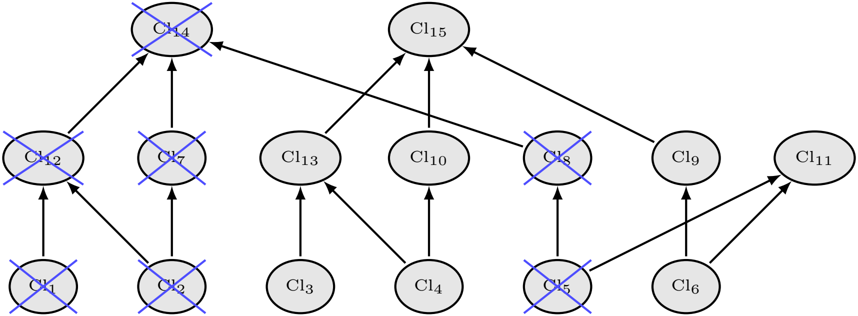

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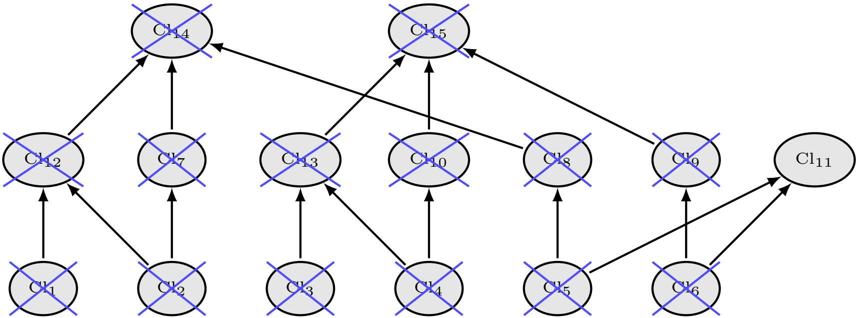

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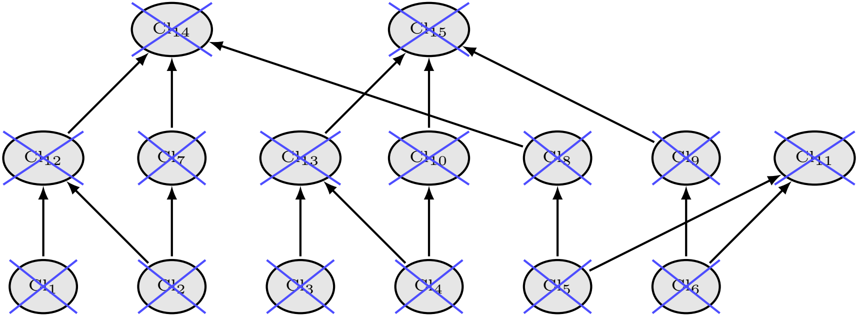


Algorithm (Without Regular Assumption)

Algorithm modified for computing $N_{\max}(\mathscr{A})$ without regular assumption:

1. Define a set $\mathscr{B}$, and initialize $\mathscr{B}$ to the empty set.
2. Arbitrarily choose a wiretap set $A \in \mathscr{A}$ ($\mathscr{A}$ is not necessary regular) that has ~~**the largest cardinality**~~ **the largest minimum cut capacity** in $\mathscr{A}$. Find the primary minimum cut between s and A, and call it CUT.
3. Partition the edge set E into two disjoint subsets: E_{CUT} and $E^c_{\mathrm{CUT}} \triangleq E \setminus E_{\mathrm{CUT}}$, where E_{CUT} is the set of the edges reachable from the source node s upon deleting the edges in CUT.
4. Remove all the wiretap sets in $\mathscr{A}$ that are subsets of E^c_{CUT} and add the primary minimum cut CUT to $\mathscr{B}$.
5. Repeat Steps 2) to 4) until $\mathscr{A}$ is empty and output $\mathscr{B}$, where $N_{\max}(\mathscr{A}) = |\mathscr{B}|$.

Algorithm (Without Regular Assumption)

However,

- This modification requires pre-computing the minimum cut capacity of every wiretap set in $\mathscr{A}$.
- This will significantly increase the computational complexity of the algorithm when $|\mathscr{A}|$ is large.

Algorithm (Without Regular Assumption)

Algorithm II modified for computing $N_{\max}(\mathscr{A})$ without regular assumption:

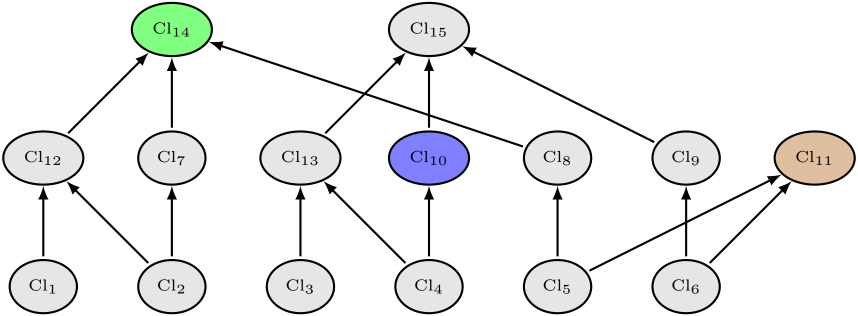
1. Define a set $\mathscr{B}$, and initialize $\mathscr{B}$ to the empty set.
2. Arbitrarily choose a wiretap set $A \in \mathscr{A}$ ($\mathscr{A}$ is not necessary regular) that has the largest cardinality in $\mathscr{A}$. Find the primary minimum cut between s and A, and call it CUT.
3. Partition the edge set E into two disjoint subsets: E_{CUT} and $E^c_{\mathrm{CUT}} \triangleq E \setminus E_{\mathrm{CUT}}$, where E_{CUT} is the set of the edges reachable from the source node s upon deleting the edges in CUT.
4. Remove ~~**all the wiretap sets in $\mathscr{A}$**~~ **all the wiretap or edge sets in $\mathscr{A} \cup \mathscr{B}$** that are subsets of E^c_{CUT}. Add the primary minimum cut CUT to $\mathscr{B}$.
5. Repeat Steps 2) to 4) until $\mathscr{A}$ is empty and output $\mathscr{B}$, where $N_{\max}(\mathscr{A}) = |\mathscr{B}|$.

Verification of Modified Algorithm II

In Step 4), CUT_A is added to $\mathscr{B}$.

- If A has the largest minimum cut capacity in $\mathscr{A}$ (e.g. Cl_{14}), then CUT_A will stay in $\mathscr{B}$ until the algorithm terminates.
- If A does not have the largest minimum cut capacity in $\mathscr{A}$,
  1. if A belongs to a maximal equivalence class (e.g. Cl_{11}), then CUT_A will stay in $\mathscr{B}$ until the algorithm terminates;
  2. otherwise (e.g. Cl_{10}), CUT_A will eventually be replaced by a primary minimum cut of a maximal equivalence class Cl with $\mathrm{Cl} \succ \mathrm{Cl}(A)$.
- Algorithm II computes the minimum cut capacity **at most** $N(\mathscr{A})$ times (instead of exactly $|\mathscr{A}|$ times).

Example (Cont.)



Algorithm II (Without Regular Assumption)

Algorithm 1: Algorithm for Computing $N_{\max}(\mathscr{A})$

Input: The wiretap network $(G, s, T, \mathscr{A})$, where $G = (V, E)$.

Output: $N_{\max}(\mathscr{A})$, the number of maximal equivalence classes with respect to $(G, s, T, \mathscr{A})$.

begin
  Set $\mathscr{B} = \emptyset$;
  while $\mathscr{A} \neq \emptyset$ **do**
    choose a wiretap set A of the largest cardinality in $\mathscr{A}$;
    find the primary minimum cut CUT of A;
    partition E into two parts E_{CUT} and $E^c_{\mathrm{CUT}} = E \setminus E_{\mathrm{CUT}}$;
    for each $B \in \mathscr{A} \cup \mathscr{B}$ **do**
      if $B \subseteq E^c_{\mathrm{CUT}}$ **then**
        remove B from $\mathscr{A}$.
      end
    end
    add CUT to $\mathscr{B}$.
  end
  Return $\mathscr{B}$. // Note that $|\mathscr{B}| = N_{\max}(\mathscr{A})$.
end

Line 5: Edge Partition

- **Line 5 in Algorithm II** can be implemented efficiently by slightly modifying existing search algorithms on directed graphs.
- The complexity is in $\mathcal{O}(|E_{\mathrm{CUT}}|)$ time.

Algorithm for Edge Partition

Algorithm 2: Search Algorithm

```
begin
    Unmark all nodes in V;
    mark source node s;
    pred(s) := 0;    // pred(i) refers to a predecessor node of node i.
    set the edge-set SET = ∅;
    set the node-set LIST = {s};
    while LIST ≠ ∅ do
        select a node i in LIST;
        if node i is incident to an edge (i, j) such that node j is unmarked
        then
            mark node j;
            pred(j) := i;
            add node j to LIST;
            add all parallel edges leading from i to j to SET;
        else
            delete node i from LIST;
        end
    end
    Return the edge-set SET.
end
```

Line 4: Finding Primary Minimum Cut

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- Instead of the primary minimum cut between s and an edge set A, we consider the primary minimum cut between s and a sink node t.
- Let f be a maximal flow from s to t. Then f can be decomposed into $n(=\mathrm{mincut}(s,t))$ edge-disjoint paths $P_1, P_2, \cdots, P_n$ from s to t such that for every edge e,

$$f(e) = \begin{cases} 1, & e \in P_i \text{ for some } 1 \le i \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Algorithm for Finding Primary Minimum Cut

Algorithm 3: Algorithm for Finding the Primary Minimum Cut

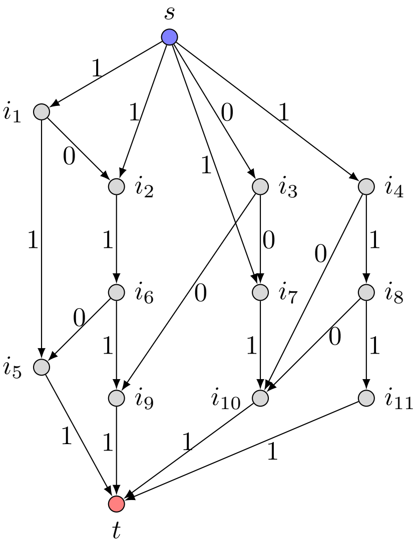
Input: An acyclic network $G = (V, E)$ with a maximal flow f from the source node s to a sink node t.

Output: The primary minimum cut between s and t.

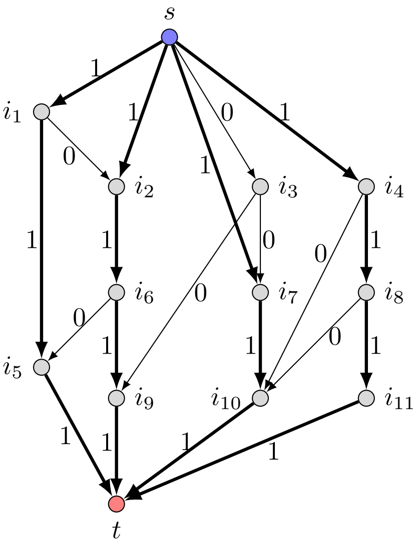
begin
- Set $S = \{s\}$;
- **for** each node $i \in S$ **do**
  - **if** **$\exists$ a node $j \in V \setminus S$ s.t. either $\exists$ a forward edge e from i to j s.t. $f(e) = 0$ or $\exists$ a reverse edge e from j to i s.t. $f(e) = 1$** **then**
    - replace S by $S \cup \{j\}$.
  - **end**
- **end**
- Return $\mathrm{CUT} = \{e : \ \mathrm{tail}(e) \in S$ and $\mathrm{head}(e) \in V \setminus S\}$.

end

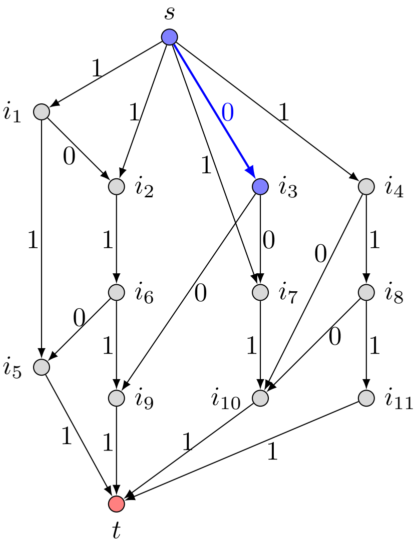
Example for Algorithm 3



Example for Algorithm 3



Example for Algorithm 3



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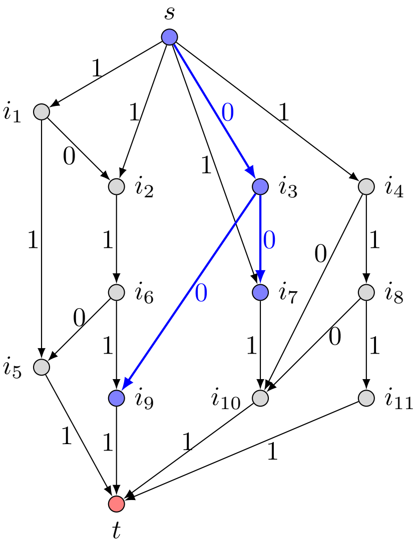

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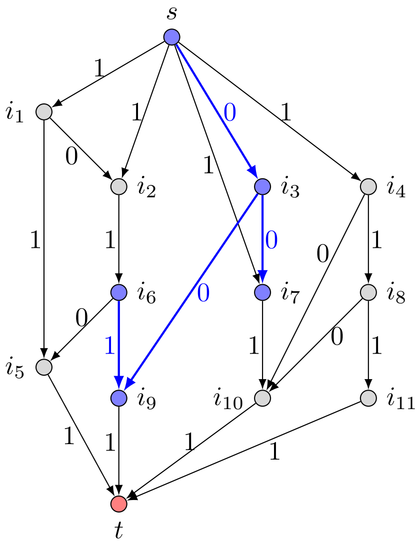

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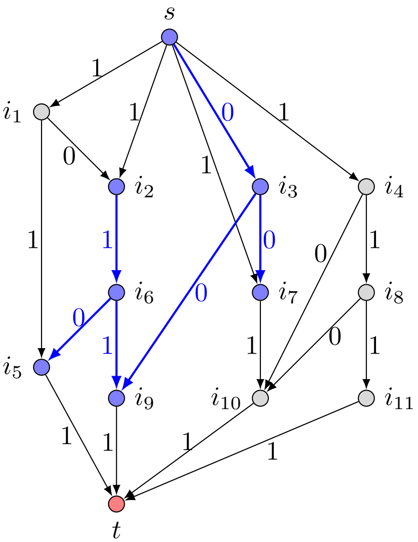

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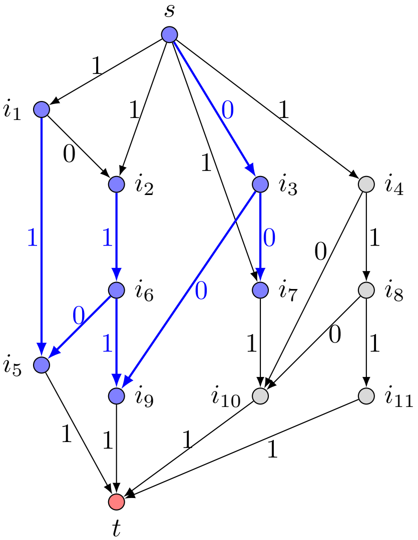

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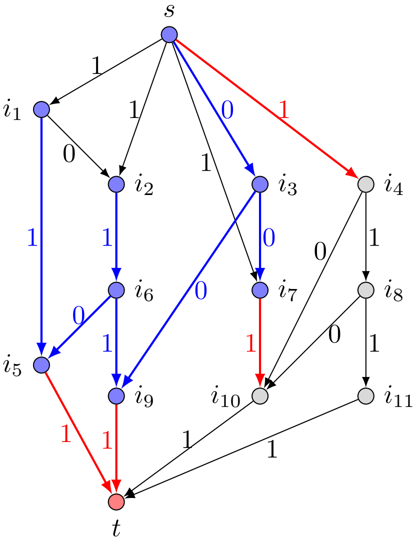

Algorithm for Finding Primary Minimum Cut

Theorem 10

The output edge set CUT *of Algorithm 3 is the primary minimum cut between* s *and* t*.*

- The complexity of Algorithm 3 does not exceed $\mathcal{O}(|E|)$ time.

Outline

1. Preliminaries
   - Secure Network Coding
   - Alphabet Size Problem
2. A New Lower Bound on Required Alphabet Size
3. Efficient Algorithm for Computing the Lower Bound
   - Primary Minimum Cut
   - Algorithm
4. Conclusion Remarks

Concluding Remarks

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- Our lower bound is applicable to both linear and non-linear secure network codes.

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- Our lower bound is applicable to both linear and non-linear secure network codes.
- Many proofs are non-trivial, involving some new techniques.
- Whether the graph theoretic approach can help solve other alphabet size problems, such as in network error correction coding.
- The concepts and results are of fundamental interest in graph theory and we expect that they will find applications in graph theory and beyond.

Happy Shannon's Centenary!!!

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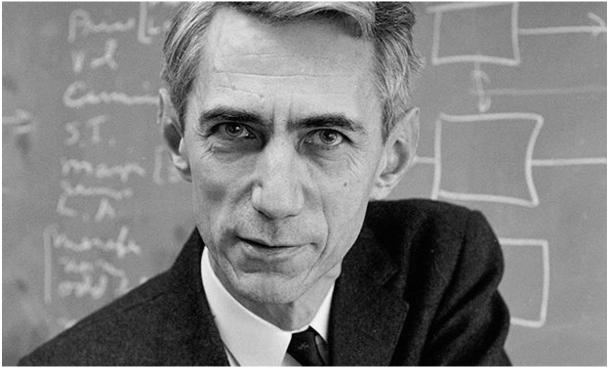

Thanks for your attention!