The Lightwave Channel: Particles or Waves

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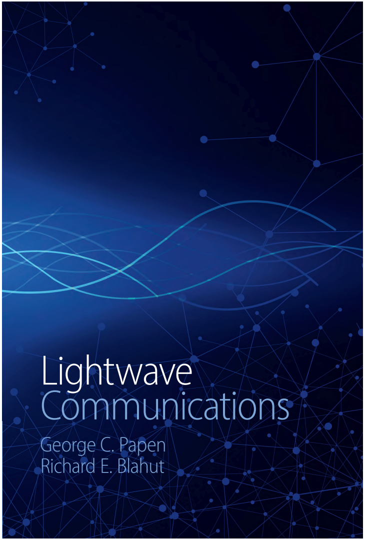
Lightwave
Communications
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Background

- Within information theory, wave and particle channels seem to be different
- Within physics, electromagnetic waves are granular at low signal levels
- Quantum theory unifies waves and particles

Background

- Information theory is a subject that exists separate from physics
- Within information theory, the Poisson transform unifies wave channels and particle channels
- Within physics, this unification would be called a semiclassical analysis

Observation

- Shannon bandlimited capacity

$$C = B \log(1 + S/N)$$

for maximum-entropy, worst-case noise.

- Energy per bit

$$E_b/N_0 \geq -1.6dB$$

- These statements are mathematically complete.
- Or are they?

Observation

- Shannon bandlimited capacity

$$C = B \log(1 + S/N)$$

- Energy per bit

$$E_b/N_0 \geq -1.6dB$$

- These statements are mathematically complete.
- These statements are not physically complete.
- These statements fail to account for granularity.

Granularity

- At 2×10^{10} Hertz (20 Gigahertz)
  One Watt equals 7.5×10^{22} photons per second
- At 2×10^{14} Hertz (1.5 microns)
  One nanowatt equals 7.5 photons per nanosecond

Information-Theoretic Channel Models

- Waveform Channel
  - Shannon (1948)
- Particle Channel
  - Many authors
- Wave/Particle Channel
  - Gordon-(Forney) Conjecture (1964)
  - Gordon formula (1962)
  - Blahut-Papen (2018)
- Quantum Channel
  - Von Nuemann (1932)
  - Holevo Bound (1972)
  - Phase Sensitive Channel — many authors (2014)

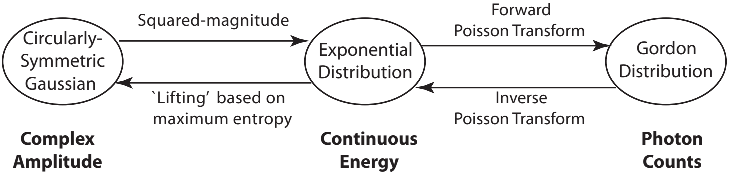
The Physics of Photons and Waves

- "Particle" and "complex baseband signal" have physical meaning using instead the terms "photon" and "wave. "
  - Photons are each associated with a constant E called the *energy*.
  - Waves are associated with a constant f called the *carrier frequency*.
- These constants are related by $E = hf$ where the scaling term h is called Planck's constant.

Objective

- Relate the channel capacity of a discrete “particle” channel to the channel capacity of a continuous “complex-baseband” channel.
- Provide a common framework to give the capacity of Poisson channel in a small-signal regime and the Shannon capacity in a large signal regime.
- Shannon capacity should be a large-signal emergent fluid model from the capacity of a particle stream.

Approach - Relate Continuous and Discrete Maximum-Entropy Distributions



Maximum Entropy Distribution for Particles

- The probability constraints are that $\sum_{\mathrm{m}=0}^{\infty} p_{\underline{\mathrm{m}}}(\mathrm{m}) = 1$, and the p_m are nonnegative.
- The maximization is constrained by a finite mean $E[m] = \mathrm{M}$.

Theorem

The maximum-entropy probability mass function is

$$p_{\underline{\mathrm{m}}}(\mathrm{m}) = \frac{1}{1+\mathrm{M}}\left(\frac{\mathrm{M}}{1+\mathrm{M}}\right)^{\mathrm{m}} \qquad \mathrm{m} = 0, 1, \ldots.$$

Proof.

This is a standard maximization using Lagrange multipliers. □

Note: This is not a Poisson distribution.

The Gordon Distribution

- This probability mass function is a geometric probability mass function called the *Gordon distribution*
- The entropy of the Gordon distribution is

$$H = \log_e(1 + \mathrm{M}) + \mathrm{M}\log_{\mathrm{e}}(1 + 1/\mathrm{M})$$

- The two terms suggest the wave and particle nature of a signal.
  - When M is large $H \approx \log_e(1 + \mathrm{M})$. This is the entropy of a continuous exponential probability density function with mean M.
  - When M is is small $H \approx M - M\log_e \mathrm{M}$. This is the small-signal expansion of the entropy of a Poisson probability mass function with mean M.
- Two limiting forms of the Gordon distribution reveal the particle and wave properties.

Special Distributions

- Probability density function
  - Maximum entropy — Real or complex gaussian
  - Convolution invariance — Real or complex gaussian
- Probability mass function
  - Maximum entropy — Gordon distribution
  - Convolution invariance — Poisson distribution

The Poisson Transform

- Let $f_{\underline{\mathsf{E}}}(\mathsf{E})$ be any continuous probability density function on the nonnegative reals. Then

$$p_{\underline{\mathsf{m}}}(\mathsf{m}) = \int_0^\infty \frac{\mathsf{E}^\mathsf{m}}{\mathsf{m}!} e^{-\mathsf{E}} f_{\underline{\mathsf{E}}}(\mathsf{E}) \mathrm{d}\mathsf{E},$$

  is a probability mass function on the nonnegative integers.
- The inverse Poisson transform maps probability mass functions to real probability density functions.

Properties of the Poisson Transform

- The Poisson transform (and its inverse) reveal the parallel roles for the wave model and the particle model of a signal.
- By analogy with the symbolic expression for the Fourier transform

$$s(t) \quad \longleftrightarrow \quad S(f)$$

  the Poisson transform is expressed symbolically as

$$f(x) \quad \leftrightsquigarrow \quad p(m).$$

Examples

- The Poisson distribution is the Poisson transform of a Dirac delta function.

$$\delta(E - M) \quad \leftsquigarrow\!\!\rightsquigarrow \quad \frac{\mathsf{M}^{\mathsf{m}}}{\mathsf{m}!} e^{-\mathsf{M}}.$$

- The Gordon distribution is the Poisson transform of a maximum-entropy exponential probability density function.

$$e^{-E/M} \quad \leftsquigarrow\!\!\rightsquigarrow \quad \frac{1}{1+\mathsf{M}} \left(\frac{\mathsf{M}}{1+\mathsf{M}} \right)^{\mathsf{m}}$$

Composite Distribution formed by the Poisson Transform

- When viewed as a Poisson transform, the Gordon distribution is the composite of:
  - The uncertainty caused by the random arrival times of particles,
  - Maximum statistical uncertainty expressed by an exponential distribution.
- The composite effect determined by
  - Considering the effect of the channel (always present).
  - Overlying maximum-entropy statistical uncertainty using the Poisson transform to give the Gordon distribution.

Lifting the Energy to the Complex Amplitude

- The square of a constant complex amplitude over a finite-time interval T is the energy E.
- The square-root of the maximum-entropy exponential distribution is not a maximum-entropy distribution
- This failure is remedied by the assertion that the square-root of the energy is complex.
- Equivalently, the energy is the sum of two squared terms, not one.

Position and Momentum

- A particle is described by both a position and a momentum and has both potential energy and kinetic energy.
- An approach that does not explicitly account for these two degrees of freedom of the particle fails to generate a maximum-entropy distribution.

Inverting the Exponential Distribution

- $x + \mathrm{i}\mathrm{y}$ is gaussian $\longrightarrow x^2 + y^2$ is exponential
- What is square root of exponential?
  - $\sqrt{E}$ is not maximum entropy
  - $\sqrt{E}$ does not convolve
- Energy must be expressed as the complex factorization

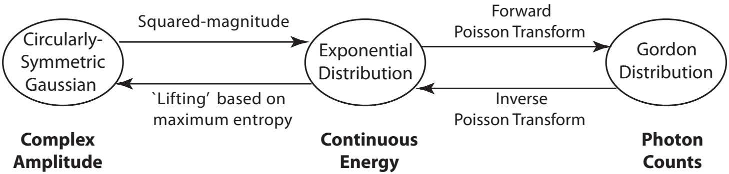
  $$E = (x + \mathrm{i}y)(x - \mathrm{i}y)$$

  or

  $$E = x^2 + y^2.$$

- This expression, hinted at by the form of the composite Gordon distribution, is an unavoidable consequence of respect for the maximum entropy principle.
- The maximum-entropy distribution of the complex amplitude $A = A_I + \mathrm{i}A_Q = |A|e^{\mathrm{i}\phi}$ is a circularly-symmetric gaussian.

Summary



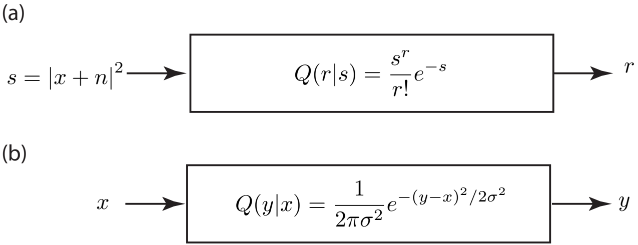
- A unified information-theoretic framework for both wave channels and particle channels
  - Consistent with quantum theory.
  - Semiclassical abridgement of quantum theory.

The Additive Noise Particle Channel

- A discrete memoryless channel is composed of contiguous equal time intervals each with a complex sinusoid
  - Average signal power constrained.
  - Independent, additive complex gaussian noise in each interval.
- The random variable $\underline{r}$ denotes the total number of received particles in an interval.
  - Poisson transform of wave intensity
- Considering only the discrete-particle aspect of the signal, the random received signal $\underline{r}$ is the sum of particle counts for the signal $\underline{s}$ and the independent additive noise $\underline{n}$

$$\underline{r} = \underline{s} + \underline{n}.$$

Two Channel Models



Mutual Information and Capacity

- Using $\underline{r} = \underline{s} + \underline{n}$, the mutual information $I(\underline{s}; \underline{r})$ is

$$I(\underline{s}; \underline{r}) = H(\underline{r}) - H(\underline{r}|\underline{s}) = H(\underline{s} + \underline{n}) - H(\underline{n}),$$

  where $H(\underline{r}|\underline{s})$ is equal to $H(\underline{n})$ because the signal $\underline{s}$ and the noise $\underline{n}$ are independent.
- The conditional probability $p(\underline{r}|\underline{s})$ that determines the entropy $H(\underline{r})$ is a conditional Poisson distribution

$$\mathrm{C} = \max I(\underline{s}; \underline{r}),$$

Maximizing the Mutual Information for the Particle Channel

- Distribution $p(\mathsf{r})$ maximizing entropy $H(\underline{\mathsf{r}})$ is the Gordon distribution given by
$$p(\mathsf{r}) = \frac{1}{1+\mathsf{Z}}\left(\frac{\mathsf{Z}}{1+\mathsf{Z}}\right)^{\mathsf{r}},$$
with entropy $H(\underline{\mathsf{r}}) = (1+\mathsf{Z})\log(1+\mathsf{Z}) - \mathsf{Z}\log\mathsf{Z}$ where Z is the expected number of received particles $\mathsf{E}+\mathsf{N}_0$.
- Capacity would be achieved for a prior $p(\mathsf{s})$ such that $p(\mathsf{s}+\mathsf{n})$ is a maximum-entropy Gordon distribution with $\mathsf{Z}=\mathsf{E}+\mathsf{N}_0$.

The Channel Capacity

- When $\underline{r}$ and $\underline{n}$ are each Poisson, and $\underline{r} = \underline{s} + \underline{n}$, then $\underline{s}$ must be Poisson as well.
- When the encoder generates an input distribution for the continuous signal energy that is the maximum-entropy exponential distribution, the corresponding discrete distribution for the number of transmitted particles is a Gordon distribution.
- Mean signal E is the difference between the mean number of received particles Z and the mean number of noise particles N_0 added by the channel.
- Using $\mathsf{Z} = \mathsf{E} + \mathsf{N}_0$, the maximum entropy $H(\underline{r})$ at the output of the channel is

$$H(\underline{r}) = g(\mathsf{Z}) \quad = \quad g(\mathsf{E} + \mathsf{N}_0).$$

The Channel Capacity

- Using the precceding expression and $H(\underline{\mathsf{n}})$, the capacity of the channel is

$$\mathsf{C} = H(\underline{\mathsf{r}}) - H(\underline{\mathsf{n}}) = g(\mathsf{E} + \mathsf{N}_0) - g(\mathsf{N}_0),$$

- This is the single-letter capacity C of the noisy time-discrete particle channel in units of bits/symbol.

$$\boxed{\mathsf{C} = \log_2\left(1 + \frac{\mathsf{E}}{1 + \mathsf{N}_0}\right) + (\mathsf{E} + \mathsf{N}_0)\log_2\left(1 + \frac{1}{\mathsf{E} + \mathsf{N}_0}\right) - \mathsf{N}_0 \log_2\left(1 + \frac{1}{\mathsf{N}_0}\right)}$$

- E is the mean number of signal counts
  - $E = \mathsf{E}hf$ is the mean signal energy
- N_0 is the mean number of noise counts
  - $N_0 = \mathsf{N}_0 hf$ is the mean noise energy

Gordon Noiseless Capacity Formula

The bandlimited capacity $\mathcal{C}$, in bits per second, of a *noiseless* particle channel with time-averaged particle arrival rate R, in bits per second,

$$\mathcal{C} \quad = \quad B \log_2 \left(1 + \frac{\mathsf{R}}{B}\right) + \mathsf{R} \log_2 \left(1 + \frac{B}{\mathsf{R}}\right)$$

with B in Hertz.

The Shannon Capacity

- The Shannon capacity based on waves can be derived directly from the particle capacity.
- When both E and N_0 are much larger than one, the entropy of a Gordon distribution $g(x)$ approaches the entropy of an exponential distribution.
- Replace mean particle counts E and N_0 by mean continuous energy E and N_0
- Replacing $g(\mathrm{m})$ in by $1 + \log x$ gives

$$\begin{aligned} \mathsf{C} &= H(\underline{\mathrm{r}}) - H(\underline{\mathrm{n}}) \\ &= 1 + \log(E + N_0) - \left(1 + \log N_0\right) \\ &= \log_2\left(1 + \frac{E}{N_0}\right) \quad \text{bits per symbol} \end{aligned}$$

which is the single-letter capacity based on waves.

Comments and Conclusion

- The Poisson transform (lifted) provides a unified framework for wave channels and particle channels
- The duality of channel capacity of particle and wave channels is one example

Information Theory Hierarchy

1. Classical information theory
   (Shannon, *et al*)

2. Semiclassical information theory
   (Gordon, Mandel, Wolf, Forney, *et al*)

3. Quantum information theory
   (von Neumann, Holevo, Shapiro, *et al*)