Two Results on Vector Gaussian Multi-Terminal Source Coding

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Outline

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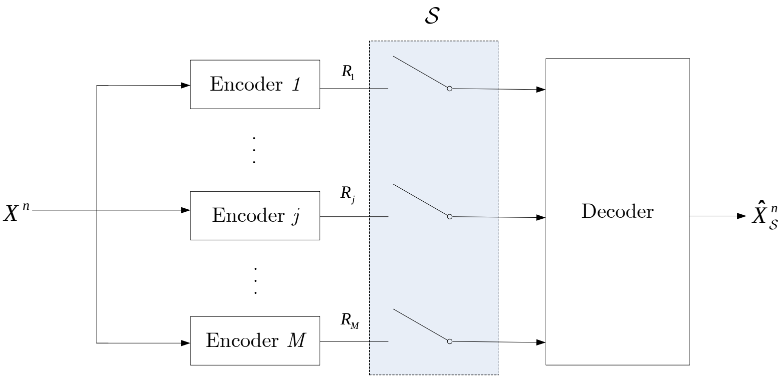
   - Problem Formulation
   - A Single-Letter Lower Bound for Sum Rate
   - Vector Gaussian Case

Outline

Multiple Description Coding



Multiple Description Coding

- Source: i.i.d. process $\{X(t)\}_{t=1}^{\infty}$ with marginal distribution $p(x)$ over alphabet $\mathcal{X}$.
- Encoding function:

$$\varphi_i^{(n)} : \mathcal{X}^n \to \mathcal{C}_i^{(n)}, \quad i = 1, \cdots, M,$$

$$\frac{1}{n} \log |\mathcal{C}_i^{(n)}| \leq R_i, \quad i = 1, \cdots M.$$

- Decoding function:

$$\psi_{\mathcal{S}}^{(n)} : \prod_{i \in \mathcal{S}} \mathcal{C}_i^{(n)} \to \hat{\mathcal{X}}^n, \quad \mathcal{S} \in 2_+^{\{1, \cdots, M\}},$$

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[d(X(t), \hat{X}_{\mathcal{S}}(t))] \leq d_{\mathcal{S}}, \quad \mathcal{S} \in 2_+^{\{1, \cdots, M\}},$$

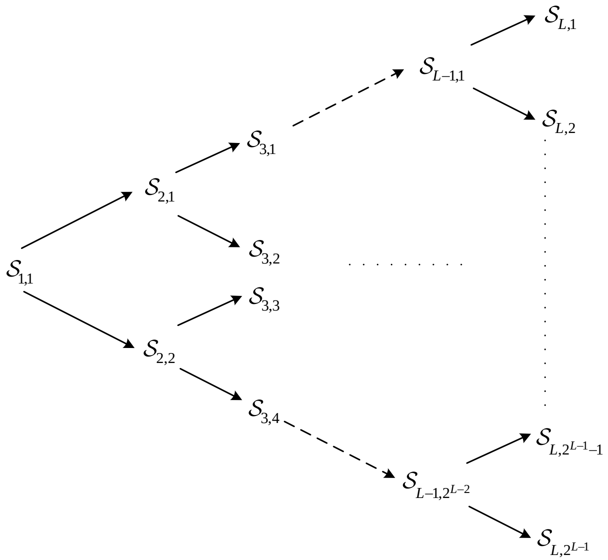
Tree-Structured Distortions

Definition

We say a set $\mathcal{T} \subseteq 2_{+}^{\{1,\cdots,M\}}$ has a tree structure if, for any $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{T}$, one of the following statements is true:

- $\mathcal{S}_1 \subseteq \mathcal{S}_2$,
- $\mathcal{S}_2 \subseteq \mathcal{S}_1$,
- $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

Tree-Structured Distortions



Tree-Structured Distortions

$$\mathcal{T} = \{\mathcal{S}_{k,i} : k = 1, \cdots, L; i = 1, \cdots, 2^{k-1}\},$$
$$\mathcal{S}_{k,i} = \left\{ j : \frac{2^L(i-1)}{2} < j \leq \frac{2^L i}{2^k} \right\}.$$

- Two-description-three-distortion problem (L. Ozarow): $\mathcal{T} = \{\{1,2\},\{1\},\{2\}\}$,
- Individual and central distortion constraints (J. Chen): $\mathcal{T} = \{\{1,\cdots,M\},\{1\},\cdots,\{M\}\}$,
- A three-level-four-description example:

$$\begin{aligned}\mathcal{T} =& \{\{1,2,3,4\},\{1,2\},\{3,4\},\{1\},\{2\},\{3\},\{4\}\} \\ =& \{\mathcal{S}_{1,1},\mathcal{S}_{2,1},\mathcal{S}_{2,2},\mathcal{S}_{3,1},\mathcal{S}_{3,2},\mathcal{S}_{3,3},\mathcal{S}_{3,4}\}\end{aligned}$$

A Single-Letter Lower Bound for Sum Rate

- $\mathcal{P}$ is the set of conditional distributions $p(\underline{z}|x)$ such that:

$$p(\underline{z}|x) = p(z_{1,1}|x)p(z_{2,1}, z_{2,2}|z_{1,1})$$

  where $\underline{z} = (z_{1,1}, z_{2,1}, z_{2,2})$.

- $\mathcal{P}(\underline{d})$ is the set of conditional distributions $p(\underline{\hat{x}}|x)$ such that the induced $p(x)p(\underline{\hat{x}}|x)$ satisfies

$$\mathbb{E}[d(X, \hat{X}_{\mathcal{S}_{k,i}})] \leq d_{\mathcal{S}_{k,i}}, \quad k = 1, 2, 3; i = 1, \cdots, 2^{k-1},$$

  where $\underline{d} = (d_{\mathcal{S}_{k,i}})_{k=1,2,3;i=1,\cdots,2^{k-1}}$ and $\underline{\hat{x}} = (\hat{x}_{\mathcal{S}_{k,i}})_{k=1,2,3;i=1,\cdots,2^{k-1}}$.

A Single-Letter Lower Bound for Sum Rate

Theorem

The sum rate $R(\underline{d})$ *is defined as*

$$R(\underline{d}) = \inf_{(R_1,R_2,R_3,R_4)\in\mathcal{R}(\underline{d})} R_1 + R_2 + R_3 + R_4,$$

which is lower bounded by

$$\begin{aligned} R(\underline{d}) \geq \inf_{p(\hat{x}|x)\in\mathcal{P}(\underline{d})} \sup_{p(\underline{z}|x)\in\mathcal{P}} & I(X;\hat{X}_{\mathcal{S}_{1,1}}|Z_{1,1}) \\ & + I(Z_{1,1};\hat{X}_{\mathcal{S}_{2,1}}|Z_{2,1}) + I(Z_{1,1};\hat{X}_{\mathcal{S}_{2,2}}|Z_{2,2}) \\ & + I(Z_{2,1};\hat{X}_{\mathcal{S}_{3,1}}) + I(Z_{2,1};\hat{X}_{\mathcal{S}_{3,2}}) \\ & + I(Z_{2,2};\hat{X}_{\mathcal{S}_{3,3}}) + I(Z_{2,2};\hat{X}_{\mathcal{S}_{3,4}}). \end{aligned}$$

where $p(\underline{z},x,\underline{\hat{x}}) = p(\underline{z}|x)p(x)p(\underline{\hat{x}}|x)$.

Proof

$$
\begin{aligned}
&\log|\mathcal{C}_1^{(n)}| + \log|\mathcal{C}_2^{(n)}| + \log|\mathcal{C}_3^{(n)}| + \log|\mathcal{C}_4^{(n)}| \\
\geq & H(\varphi_1^{(n)}) + H(\varphi_2^{(n)}) + H(\varphi_3^{(n)}) + H(\varphi_4^{(n)}) \\
= & H(\varphi_1^{(n)}, \varphi_2^{(n)}) + H(\varphi_3^{(n)}, \varphi_4^{(n)}) + I(\varphi_1^{(n)}; \varphi_2^{(n)}) + I(\varphi_3^{(n)}; \varphi_4^{(n)}) \\
= & H(\varphi_1^{(n)}, \varphi_2^{(n)}, \varphi_3^{(n)}, \varphi_4^{(n)}) + I(\varphi_1^{(n)}; \varphi_2^{(n)}) + I(\varphi_3^{(n)}; \varphi_4^{(n)}) \\
&+ I(\varphi_1^{(n)}, \varphi_2^{(n)}; \varphi_3^{(n)}, \varphi_4^{(n)})
\end{aligned}
$$

Proof

- Introducing auxiliary remote source $Z_{2,1}^n$

$$\begin{aligned} & I(\varphi_1^{(n)};\varphi_2^{(n)}) \\ & = I(Z_{2,1}^n;\varphi_1^n) + I(Z_{2,1}^n;\varphi_2^{(n)}) - I(Z_{2,1}^n;\varphi_1^{(n)},\varphi_2^{(n)}) \\ & \quad + I(\varphi_1^n;\varphi_2^{(n)}|Z_{2,1}^n) \\ & \geq I(Z_{2,1}^n;\varphi_1^n) + I(Z_{2,1}^n;\varphi_2^{(n)}) - I(Z_{2,1}^n;\varphi_1^{(n)},\varphi_2^{(n)}). \end{aligned}$$

- Similarly introducing remote sources

$$\begin{aligned} & Z_{2,2}^n \text{ to } I(\varphi_3^{(n)};\varphi_4^{(n)}), \\ & Z_{1,1}^n \text{ to } I(\varphi_1^{(n)},\varphi_2^{(n)};\varphi_3^{(n)},\varphi_4^{(n)}). \end{aligned}$$

Proof

$$
\begin{aligned}
&\log|\mathcal{C}_1^{(n)}| + \log|\mathcal{C}_2^{(n)}| + \log|\mathcal{C}_3^{(n)}| + \log|\mathcal{C}_4^{(n)}| \\
\geq & I(X^n;\varphi_1^{(n)},\varphi_2^{(n)},\varphi_3^{(n)},\varphi_4^{(n)}) - I(Z_{1,1}^n;\varphi_1^{(n)},\varphi_2^{(n)},\varphi_3^{(n)},\varphi_4^{(n)}) \\
&+ I(Z_{1,1}^n;\varphi_1^{(n)},\varphi_2^{(n)}) - I(Z_{2,1}^n;\varphi_1^{(n)},\varphi_2^{(n)}) \\
&+ I(Z_{1,1}^n;\varphi_3^{(n)},\varphi_4^{(n)}) - I(Z_{2,2}^n;\varphi_3^{(n)},\varphi_4^{(n)}) \\
&+ I(Z_{2,1}^n;\varphi_1^{(n)}) \\
&+ I(Z_{2,1}^n;\varphi_2^{(n)}) \\
&+ I(Z_{2,2}^n;\varphi_3^{(n)}) \\
&+ I(Z_{2,2}^n;\varphi_4^{(n)})
\end{aligned}
$$

Proof

- Single letterlization of each term by time sharing random variable: T

e.g.,

$$\begin{aligned}
& I(Z_{1,1}^n; \varphi_1^{(n)}, \varphi_2^{(n)}) - I(Z_{2,1}^n; \varphi_1^{(n)}, \varphi_2^{(n)}) \\
& = I(Z_{1,1}^n; \varphi_1^{(n)}, \varphi_2^{(n)} | Z_{2,1}^n) \quad (X^n \rightarrow Z_{1,1}^n \rightarrow Z_{2,1}^n) \\
& \geq I(Z_{1,1}^n; \hat{X}_{\mathcal{S}_{2,1}}^n | Z_{2,1}^n) \\
& \geq nI(Z_{1,1}(T); \hat{X}_{\mathcal{S}_{1,1}}(T) | Z_{2,1}(T)).
\end{aligned}$$

Vector Gaussian Case: Converse Theorem

- Construction of auxiliary remote sources:

$$Z_{1,1} = X + N_{1,1}, \quad Z_{2,1} = X + N_{2,1}, \quad Z_{2,2} = X + N_{2,2},$$

where

$$\mathbf{0} \preceq \mathbf{\Sigma}_{N_{1,1}} \preceq \mathbf{\Sigma}_{N_{2,1}}, \mathbf{\Sigma}_{N_{2,2}}.$$

- Introducing notations

$$\mathbf{\Theta}_{1,1} = \left(\mathbf{\Sigma}_X^{-1} + \mathbf{\Sigma}_{N_{1,1}}^{-1}\right)^{-1},$$

$$\mathbf{\Theta}_{2,1} = \left(\mathbf{\Sigma}_X^{-1} + \mathbf{\Sigma}_{N_{2,1}}^{-1}\right)^{-1}, \quad \mathbf{\Theta}_{2,2} = \left(\mathbf{\Sigma}_X^{-1} + \mathbf{\Sigma}_{N_{2,2}}^{-1}\right)^{-1}.$$

$$\Rightarrow \quad \mathbf{\Sigma}_X \succeq \mathbf{\Theta}_{2,1}, \mathbf{\Theta}_{2,2} \succeq \mathbf{\Theta}_{1,1} \succeq \mathbf{0}.$$

Vector Gaussian Case: Converse Theorem

- Applying worst noise lemma to each term: e.g.

$$
\begin{aligned}
& I(X;\hat{X}_{\mathcal{S}_{1,1}}|Z_{1,1}) \\
&= I(X;\hat{X}_{\mathcal{S}_{1,1}}) - I(Z_{1,1};\hat{X}_{\mathcal{S}_{1,1}}) \\
&= \frac{1}{2}\log\left(\frac{\mathbf{\Sigma}_X}{\mathbf{\Sigma}_X + \mathbf{\Sigma}_{N_{1,1}}}\right) + I(N_{1,1}; X + N_{1,1}|\hat{X}_{\mathcal{S}_{1,1}}) \\
&\geq \frac{1}{2}\log\frac{|\mathbf{\Sigma}_X||\mathbf{D}_{\mathcal{S}_{1,1}} + \mathbf{\Sigma}_{N_{1,1}}|}{|\mathbf{D}_{\mathcal{S}_{1,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{N_{1,1}}|} \\
&= \frac{1}{2}\log\frac{|\mathbf{\Theta}_{1,1}^{-1} + \mathbf{D}_{\mathcal{S}_{1,1}}^{-1} - \mathbf{\Sigma}_X^{-1}|}{|\mathbf{\Theta}_{1,1}^{-1}|}
\end{aligned}
$$

Vector Gaussian Case: Converse Theorem

$$
\begin{aligned}
R(\underline{d}) \geq \max_{\boldsymbol{\Theta}_{1,1},\boldsymbol{\Theta}_{2,1},\boldsymbol{\Theta}_{2,2}} & \frac{1}{2}\log\frac{|\boldsymbol{\Theta}_{1,1}^{-1}+\mathbf{D}_{\mathcal{S}_{1,1}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|}{|\boldsymbol{\Theta}_{1,1}^{-1}|} \\
& + \frac{1}{2}\log\frac{|\boldsymbol{\Theta}_{1,1}^{-1}||\boldsymbol{\Theta}_{2,1}^{-1}+\mathbf{D}_{\mathcal{S}_{2,1}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|}{|\boldsymbol{\Theta}_{2,1}^{-1}||\boldsymbol{\Theta}_{2,1}^{-1}+\mathbf{D}_{\mathcal{S}_{2,1}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|} \\
& + \frac{1}{2}\log\frac{|\boldsymbol{\Theta}_{1,1}^{-1}||\boldsymbol{\Theta}_{2,2}^{-1}+\mathbf{D}_{\mathcal{S}_{2,2}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|}{|\boldsymbol{\Theta}_{2,2}^{-1}||\boldsymbol{\Theta}_{2,1}^{-1}+\mathbf{D}_{\mathcal{S}_{2,2}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|} \\
& + \sum_{i=1}^{2}\frac{1}{2}\log\frac{|\boldsymbol{\Theta}_{2,1}^{-1}||\mathbf{D}_{\mathcal{S}_{3,i}}^{-1}|}{|\boldsymbol{\Sigma}_X^{-1}||\boldsymbol{\Theta}_{2,1}^{-1}+\mathbf{D}_{\mathcal{S}_{3,i}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|} \\
& + \sum_{i=3}^{4}\frac{1}{2}\log\frac{|\boldsymbol{\Theta}_{2,2}^{-1}||\mathbf{D}_{\mathcal{S}_{3,i}}^{-1}|}{|\boldsymbol{\Sigma}_X^{-1}||\boldsymbol{\Theta}_{2,2}^{-1}+\mathbf{D}_{\mathcal{S}_{3,i}}^{-1}-\boldsymbol{\Sigma}_X^{-1}|} \\
& \text{subject to} \quad \boldsymbol{\Sigma}_X \succeq \boldsymbol{\Theta}_{2,1},\boldsymbol{\Theta}_{2,2} \succeq \boldsymbol{\Theta}_{1,1} \succeq \mathbf{0}
\end{aligned}
$$

Vector Gaussian Case: Direct Theorem

$$\mathbf{\Sigma}_{\mathcal{S}_{k,i}} \triangleq (\mathbf{D}_{\mathcal{S}_{k,i}}^{-1} - \mathbf{\Sigma}_X^{-1})^{-1} \succ \mathbf{0}, \quad k = 1,2,3; i = 1,\cdots,2^{k-1}.$$

$$\begin{aligned}
\max_{\mathbf{\Theta}} \ & \frac{1}{2}\log\frac{|\mathbf{\Sigma}_{\mathbf{X}} + \mathbf{\Sigma}_{\mathcal{S}_{1,1}}|}{|\mathbf{\Sigma}_{\mathcal{S}_{1,1}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{1,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{2,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{2,2}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{1,1}}||\mathbf{\Theta}_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}}||\mathbf{\Theta}_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,2}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,2}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{2,1}}||\mathbf{\Theta}_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,1}}||\mathbf{\Theta}_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,2}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{2,2}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,3}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,4}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{2,2}}||\mathbf{\Theta}_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,3}}||\mathbf{\Theta}_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,4}}|}
\end{aligned}$$

subject to $\quad \mathbf{\Sigma}_X \succeq \mathbf{\Theta}_{2,1}, \mathbf{\Theta}_{2,2} \succeq \mathbf{\Theta}_{1,1} \succeq \mathbf{0}$

Vector Gaussian Case: Direct Theorem

KKT conditions

Let $(\mathbf{\Theta}^*_{1,1}, \mathbf{\Theta}^*_{2,1}, \mathbf{\Theta}^*_{2,2})$ be an optimizer, then there exists $(\mathbf{M}_{1,1}, \mathbf{M}_{2,1}, \mathbf{M}_{2,2}, \mathbf{M}_{3,1}, \mathbf{M}_{3,2})$ such that:

$$\begin{aligned}
(\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}})^{-1} + \mathbf{M}_{2,1} + (\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,2}})^{-1} + \mathbf{M}_{2,2} \\
= (\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{1,1}})^{-1} + \mathbf{M}_{1,1} \quad (1.1) \\
(\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,1}})^{-1} + \mathbf{M}_{3,1} + (\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,2}})^{-1} \\
= (\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}})^{-1} + \mathbf{M}_{2,1} \quad (1.2) \\
(\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,3}})^{-1} + \mathbf{M}_{3,2} + (\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,4}})^{-1} \\
= (\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{2,2}})^{-1} + \mathbf{M}_{2,2}, \quad (1.3)
\end{aligned}$$

Vector Gaussian Case: Direct Theorem

$$\mathbf{\Theta}^*_{1,1}\mathbf{M}_{1,1} = \mathbf{0}, \quad (1.4)$$

$$(\mathbf{\Theta}^*_{2,1} - \mathbf{\Theta}^*_{1,1})\mathbf{M}_{2,1} = \mathbf{0}, \qquad (\mathbf{\Theta}^*_{2,2} - \mathbf{\Theta}^*_{1,1})\mathbf{M}_{2,2} = \mathbf{0}, \quad (1.5)$$

$$(\mathbf{\Sigma}_X - \mathbf{\Theta}^*_{2,1})\mathbf{M}_{3,1} = \mathbf{0}, \qquad (\mathbf{\Sigma}_X - \mathbf{\Theta}^*_{2,2})\mathbf{M}_{3,2} = \mathbf{0}. \quad (1.6)$$

Vector Gaussian Case: Direct Theorem

Distortion enhancement: define

$$(\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}})^{-1} = (\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{1,1}})^{-1} + \mathbf{M}_{1,1}$$
$$(\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,1}})^{-1} = (\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}})^{-1} + \mathbf{M}_{2,1}$$
$$(\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,2}})^{-1} = (\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\mathcal{S}_{2,2}})^{-1} + \mathbf{M}_{2,2}$$
$$(\mathbf{\Theta}^*_{2,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,1}})^{-1} = (\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,1}})^{-1} + \mathbf{M}_{3,1}$$
$$(\mathbf{\Theta}^*_{2,2} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,3}})^{-1} = (\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,3}})^{-1} + \mathbf{M}_{3,2}$$

Along the same line of channel enhancement argument (H. Weingarten et al), the optimization lower bound can be written in another way.

Vector Gaussian Case: Direct Theorem

$$
\begin{aligned}
R(\underline{d}) \geq & \frac{1}{2}\log\frac{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{1,1}}|}{|\mathbf{\Sigma}_{\tilde{\mathcal{S}}_{1,1}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{1,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,2}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{1,1}}||\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,1}}||\mathbf{\Theta}^*_{1,1} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,2}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{3,1}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,2}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,1}}||\mathbf{\Theta}^*{}_{2,1} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{3,1}}||\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,2}}|} \\
& + \frac{1}{2}\log\frac{|\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,2}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{3,3}}||\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\mathcal{S}_{3,4}}|}{|\mathbf{\Sigma}_X + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{2,2}}||\mathbf{\Theta}^*{}_{2,2} + \mathbf{\Sigma}_{\tilde{\mathcal{S}}_{3,3}}||\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,4}}|}
\end{aligned}
$$

Vector Gaussian Case: Direct Theorem

$$(\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,1}})^{-1} + (\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,2}})^{-1} = (\mathbf{\Theta}^*_{1,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}})^{-1} \tag{1.7}$$

$$(\mathbf{\Theta}^*_{2,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,1}})^{-1} + (\mathbf{\Theta}^*_{2,1} + \mathbf{\Sigma}_{\mathcal{S}_{3,2}})^{-1} = (\mathbf{\Theta}^*_{2,1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,1}})^{-1} \tag{1.8}$$

$$(\mathbf{\Theta}^*_{2,2} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,3}})^{-1} + (\mathbf{\Theta}^*_{2,2} + \mathbf{\Sigma}_{\mathcal{S}_{3,4}})^{-1} = (\mathbf{\Theta}^*_{2,2} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,2}})^{-1}, \tag{1.9}$$

Vector Gaussian Case: Direct Theorem

Define independent zero means Gaussian random vectors $W_{1,1}, (W_{2,1}, W_{2,2}), (W_{3,1}, W_{3,2}), (W_{3,3}, W_{3,4})$ with covariance:

$$\mathrm{cov}(W_{1,1}) = \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}}$$

$$\mathrm{cov}(W_{2,1}, W_{2,2}) = \begin{pmatrix} \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,1}} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}} & -\mathbf{\Theta}^*_{1,1} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}} \\ -\mathbf{\Theta}^*_{1,1} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}} & \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,2}} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{1,1}} \end{pmatrix}$$

$$\mathrm{cov}(W_{3,2i-1}, W_{3,2i}) = \begin{pmatrix} \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,2i-1}} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,i}} & -\mathbf{\Theta}^*_{2,i} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,i}} \\ -\mathbf{\Theta}^*_{2,i} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,i}} & \mathbf{\Sigma}_{\mathcal{S}_{3,2i}} - \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{3,i}} \end{pmatrix}$$

$$i = 1, 2.$$

Vector Gaussian Case: Direct Theorem

Theorem

For $k = 1, 2, 3; i = 1, \cdots, 2^{k-1}$,

- $\mathrm{cov}(W_{k,2i-1}, W_{k,2i}) \succeq \mathbf{0}$,
- *There exists* $(\mathbf{H}_{k+1,2i-1}, \mathbf{H}_{k+1,2i})$ *such that the following identities hold*

$$\mathbf{H}_{k+1,2i-1} + \mathbf{H}_{k+1,2i-1} = \mathbf{I}$$
$$\mathbf{H}_{k+1,2i-1} W_{k+1,2i-1} + \mathbf{H}_{k+1,2i} W_{k+1,2i} = \mathbf{0}.$$

Define $U_j, i = 1, 2, 3, 4$ as output of each description

$$U_1 = X + W_{1,1} + W_{2,1} + W_{3,1},$$
$$U_2 = X + W_{1,1} + W_{2,1} + W_{3,2},$$
$$U_3 = X + W_{1,1} + W_{2,2} + W_{3,3},$$
$$U_4 = X + W_{1,1} + W_{2,2} + W_{3,4}.$$

Vector Gaussian Case: Direct Theorem

Check the covariance: $\mathrm{cov}(X|U_{k,i})$, $k = 1, 2, 3; i = 1, \ldots, 2^{k-1}$.

e.g.

$$\begin{aligned}
&\mathrm{cov}(X|U_1, U_2)\\
=&\mathrm{cov}(X|X + W_{1,1} + W_{2,1} + W_{3,1}, X + W_{1,1} + W_{2,1} + W_{3,2})\\
=&\mathrm{cov}\left(X|\mathbf{H}_{3,1}(X + W_{1,1} + W_{2,1} + W_{3,1})\right.\\
&\quad\left.+\mathbf{H}_{3,2}(X + W_{1,1} + W_{2,1} + W_{3,2})\right)\\
=&\mathrm{cov}(X|X + W_{1,1} + W_{2,1})\\
=&\left(\mathbf{\Sigma}_X^{-1} + \tilde{\mathbf{\Sigma}}_{\mathcal{S}_{2,1}}^{-1}\right)^{-1} \preceq \left(\mathbf{\Sigma}_X^{-1} + \mathbf{\Sigma}_{\mathcal{S}_{2,1}}^{-1}\right)^{-1} = \mathbf{D}_{\mathcal{S}_{2,1}}
\end{aligned}$$

Vector Gaussian Case: Direct Theorem

Check the sum rate for El Gamal-Cover inner bound

$$\begin{aligned} & h(U_1)+h(U_2)+h(U_3)+h(U_4)-h(U_1,U_2,U_3,U_4|X) \\ =& I(X;U_1,U_2,U_3,U_4)+I(U_1,U_2;U_3,U_4)+ \\ & I(U_1;U_2)+I(U_3;U_4) \end{aligned}$$

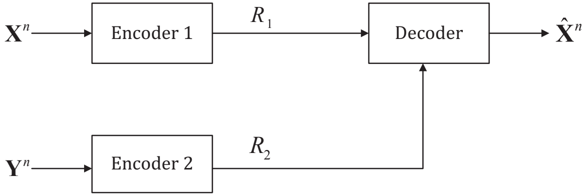
Vector Gaussian Case: Direct Theorem

Check the mutual information term: e.g. $I(U_1;U_2)$:

Write $X = \tilde{X} + \hat{X}$, where $\tilde{X}$ and $\hat{X}$ have covariance $\mathbf{\Sigma}_X - \mathbf{\Theta}^*_{2,1}$ and $\mathbf{\Theta}^*_{2,1}$.

$$\begin{aligned}
& I(U_1;U_2) \\
=& I(\tilde{X};U_1) + I(\tilde{X};U_2) - I(\tilde{X};U_1,U_2) + I(U_1;U_2|\tilde{X}) \\
=& I(\tilde{X};\tilde{X}+\hat{X}+W_{1,1}+W_{2,1}+W_{3,1}) \quad \left(\frac{1}{2}\log\frac{|\mathbf{\Sigma}_X+\mathbf{\Sigma}_{\tilde{S}_{3,1}}|}{|\mathbf{\Theta}^*_{2,1}+\mathbf{\Sigma}_{\tilde{S}_{3,1}}|}\right) \\
+& I(\tilde{X};\tilde{X}+\hat{X}+W_{1,1}+W_{2,1}+W_{3,2}) \quad \left(\frac{1}{2}\log\frac{|\mathbf{\Sigma}_X+\mathbf{\Sigma}_{\tilde{S}_{3,2}}|}{|\mathbf{\Theta}^*_{2,1}+\mathbf{\Sigma}_{\tilde{S}_{3,2}}|}\right) \\
-& I(\tilde{X};\tilde{X}+\hat{X}+W_{1,1}+W_{2,1}) \quad \left(-\frac{1}{2}\log\frac{|\mathbf{\Sigma}_X+\mathbf{\Sigma}_{\tilde{S}_{2,1}}|}{|\mathbf{\Theta}^*_{2,1}+\mathbf{\Sigma}_{\tilde{S}_{2,1}}|}\right)
\end{aligned}$$

Problem Formulation

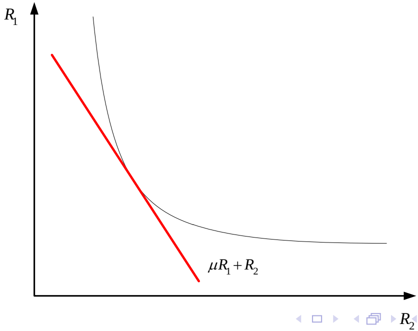


- $\{\mathbf{X}_i, \mathbf{Y}_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random vectors.
- $\mathbf{X}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{\mathbf{X}})$; $\mathbf{Y}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{\mathbf{Y}})$.
- Correlation: $\mathbf{X} = \mathbf{Y} + \mathbf{N}$, $\mathbf{N} \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{\mathbf{N}})$.

Problem Formulation

- The rate-distortion region $\mathcal{R}(\mathbf{D})$ is the closure of all achievable rate tuples (R_1, R_2) subject to distortion $\mathbf{D}$.
- Tangent hyperplane characterization: given any fixed $\mathbf{D} \succ \mathbf{0}$ and $\mu \geq 0$,

$$R(\mathbf{D}, \mu) = \inf_{(R_1, R_{2)} \in \mathcal{R}(\mathbf{D})} \mu R_1 + R_2$$



Problem Formulation

- Berger-Tung inner bound:

$$R^*(\mathbf{D},\mu) \triangleq \min_{\mathbf{B}_1,\mathbf{B}_2} \frac{\mu}{2}\log\frac{|\mathbf{K_X}-\mathbf{B}_2|}{|\mathbf{K_X}-\mathbf{B}_1-\mathbf{B}_2|}+\frac{1}{2}\log\frac{|\mathbf{K_Y}|}{|\mathbf{K_Y}-\mathbf{B}_2|}$$
$$\text{subject to } \mathbf{B}_1,\ \mathbf{B}_2 \succcurlyeq \mathbf{0},\ \text{and}$$
$$\mathbf{D} \succcurlyeq \mathbf{K_X}-\mathbf{B}_1-\mathbf{B}_2.$$

- Rahman and Wagner's theorem: $R(\mathbf{D},\mu) = R^*(\mathbf{D},\mu)$.
  "distortion projection" + *"source enhancement"*
- Our proof: Perturbation
  (*An enhanced Liu-Viswanath's extremal inequality*)

KKT Conditions for Berger-Tung inner Bound

$$\frac{\mu}{2}(\mathbf{K_X} - \mathbf{B}_1^* - \mathbf{B}_2^*)^{-1} - \mathbf{\Psi}_1 - \mathbf{\Lambda} = \mathbf{0}, \quad (2.1)$$

$$\frac{\mu}{2}(\mathbf{K_X} - \mathbf{B}_1^* - \mathbf{B}_2^*)^{-1} - \frac{\mu}{2}(\mathbf{K_X} - \mathbf{B}_2^*)^{-1} + \frac{1}{2}(\mathbf{K_Y} - \mathbf{B}_2^*)^{-1} - \mathbf{\Psi}_2 - \mathbf{\Lambda} = \mathbf{0}, \quad (2.2)$$

$$\mathbf{B}_1^*\mathbf{\Psi}_1 = \mathbf{0}, \quad (2.3)$$

$$\mathbf{B}_2^*\mathbf{\Psi}_2 = \mathbf{0}, \quad (2.4)$$

$$(\mathbf{D} - \mathbf{K_X} + \mathbf{B}_1^* + \mathbf{B}_2^*)\mathbf{\Lambda} = \mathbf{0}, \quad (2.5)$$

$$\mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Lambda} \succeq \mathbf{0}.$$

KKT Conditions for Berger-Tung inner Bound

$$\frac{\mu}{2}(\mathbf{K}_{\mathbf{X}} - \mathbf{B}_1^* - \mathbf{B}_2^*)^{-1} = \mathbf{\Psi}_1 + \mathbf{\Lambda}, \tag{2.1}$$

$$\frac{\mu}{2}(\mathbf{K}_{\mathbf{X}} - \mathbf{B}_2^*)^{-1} - \frac{1}{2}(\mathbf{K}_{\mathbf{Y}} - \mathbf{B}_2^*)^{-1} = \mathbf{\Psi}_2 + \mathbf{\Lambda}, \tag{2.2}$$

$$\mathbf{B}_1^* \mathbf{\Psi}_1 = \mathbf{0}, \tag{2.3}$$

$$\mathbf{B}_2^* \mathbf{\Psi}_2 = \mathbf{0}, \tag{2.4}$$

$$(\mathbf{D} - \mathbf{K}_{\mathbf{X}} + \mathbf{B}_1^* + \mathbf{B}_2^*)\mathbf{\Lambda} = \mathbf{0}, \tag{2.5}$$

$$\mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Lambda} \succeq \mathbf{0}.$$

The Extremal Inequality

Theorem

Let $(\mathbf{B}_1^*, \mathbf{B}_2^*)$ be the optimal solution for the optimization problem $R^*(\mathbf{D}, \mu)$. Let V be a random variable such that $\mathbf{X} \to \mathbf{Y} \to V$ forms a Markov chain. Then for every distribution of (U, V) satisfying constraints:

$$E\left[(\mathbf{X} - E[\mathbf{X}|U,V])(\mathbf{X} - E[\mathbf{X}|U,V])^T\right] \preceq \mathbf{D},$$

we have

$$\begin{aligned}
&\mu h(\mathbf{X}|V) - h(\mathbf{Y}|V) - \mu h(\mathbf{X}|U,V) \\
\geq &\frac{\mu}{2}\log|(2\pi e)(\mathbf{K_X} - \mathbf{B}_2^*)| - \frac{1}{2}\log|(2\pi e)(\mathbf{K_Y} - \mathbf{B}_2^*)| \\
&- \frac{\mu}{2}\log|(2\pi e)(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*)|.
\end{aligned}$$

The Extremal Inequality

- Covariance preserving transform: for any $\gamma \in (0,1)$,

$$\mathbf{X}_{1,\gamma} = \sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_1^G, \quad \mathbf{X}_1^G \sim \mathcal{N}(\mathbf{0}, \mathbf{K_X} - \mathbf{B_2^*} - \mathbf{B_1^*}).$$
$$\mathbf{Y}_{2,\gamma} = \sqrt{1-\gamma}\mathbf{Y} + \sqrt{\gamma}\mathbf{Y}_2^G, \quad \mathbf{Y}_2^G \sim \mathcal{N}(\mathbf{0}, \mathbf{K_Y} - \mathbf{B_2^*}).$$
$$\mathbf{X}_{2,\gamma} = \sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_2^G, \quad \mathbf{X}_2^G \sim \mathcal{N}(\mathbf{0}, \mathbf{K_X} - \mathbf{B_2^*}).$$

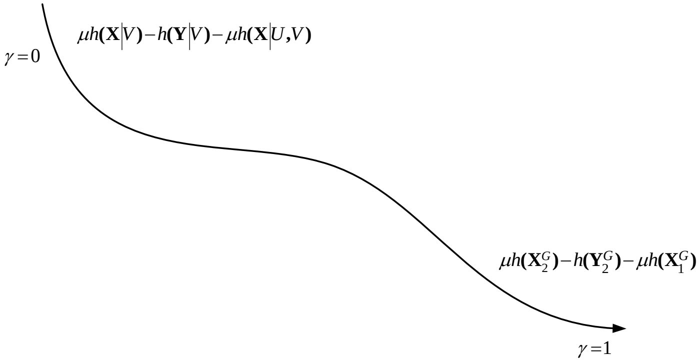
- Degraded assumptions:

$$\mathbf{X}_2^G = \mathbf{Y}_2^G + \mathbf{N}^G, \quad \mathbf{N}^G \sim \mathcal{N}(\mathbf{0}, \mathbf{K_N}),$$
$$\mathbf{X}_2^G = \mathbf{X}_1^G + \mathbf{W}^G, \quad \mathbf{W}^G \sim \mathcal{N}(\mathbf{0}, \mathbf{B_1^*})$$

- Define

$$g(\gamma) = \mu h(\mathbf{X}_{2,\gamma}|V) - h(\mathbf{Y}_{2,\gamma}|V) - \mu h(\mathbf{X}_{1,\gamma}|U,V).$$

The Extremal Inequality



- If the inequality $(1-\gamma)g'(\gamma) \leq 0$ is obtained, the extremal inequality is proved.

The Extremal Inequality

$$
\begin{aligned}
&2(1-\gamma)g'(\gamma)\\
&= \operatorname{tr}\Big\{\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}\left[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})\right.\\
&\qquad \left.-(\mathbf{K_X}-\mathbf{B_2^*})\right]\\
&\qquad -(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}\left[(\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)(\mathbf{K_Y}-\mathbf{B_2^*})\right.\\
&\qquad \left.-(\mathbf{K_Y}-\mathbf{B_2^*})\right]\\
&\quad -\Big\{\mu(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})^{-1}\left[(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})J(\mathbf{X}_{1,\gamma}|U,V)\right.\\
&\qquad \left.(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})\right]
\end{aligned}
$$

Derivation of $g(\gamma)$via de Brujin's Identity

$$
\frac{d}{d\gamma}h(\mathbf{X}+\sqrt{\gamma}\mathbf{N}|U) = \frac{1}{2}\operatorname{tr}\left\{J(\mathbf{X}+\sqrt{\gamma}\mathbf{N}|U)\right\}
$$

The Extremal Inequality

$$
\begin{aligned}
& 2(1-\gamma)g'(\gamma) \\
\leq\ & \operatorname{tr}\Big\{\mu(\mathbf{K_X}-\mathbf{B_2^*})^{-1}\big[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*}) \\
& \quad -(\mathbf{K_X}-\mathbf{B_2^*})\big] \\
& \quad -(\mathbf{K_Y}-\mathbf{B_2^*})^{-1}\big[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*}) \\
& \quad -(\mathbf{K_X}-\mathbf{B_2^*})\big] \\
& -\Big\{\mu(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})^{-1}\big[(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})J(\mathbf{X}_{1,\gamma}|V) \\
& \quad (\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_1^*}-\mathbf{B_2^*})\big]
\end{aligned}
$$

$\mathbf{X}_{2,\gamma} \overset{(d)}{=} \tilde{\mathbf{X}}_{2,\gamma} = \mathbf{Y}_{2,\gamma} + \tilde{\mathbf{N}}$, where $\tilde{\mathbf{N}} \overset{(d)}{=} \mathbf{N} \sim \mathcal{N}(\mathbf{0}, \mathbf{K_N})$.

$$
\begin{aligned}
& (\mathbf{K_Y}-\mathbf{B_2^*})J(\mathbf{Y}_{2,\gamma}|V)(\mathbf{K_Y}-\mathbf{B_2^*}) \\
\succeq\ & (\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*}) - \mathbf{K_N}.
\end{aligned}
$$

The Extremal Inequality

$$
\begin{aligned}
&(1-\gamma)g'(\gamma)\\
&\le \mathrm{tr}\,\{[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\\
&\qquad \cdot J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})+\mathbf{B}_1^*]\boldsymbol{\Psi}_1\}\\
&\quad -\mathrm{tr}\,\{[(\mathbf{K_X}-\mathbf{B_2^*})J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X}-\mathbf{B_2^*})-(\mathbf{K_X}-\mathbf{B_2^*})]\boldsymbol{\Psi_2}\}\\
&\quad -\mathrm{tr}\,\{[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\\
&\qquad\qquad -(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})]\boldsymbol{\Lambda}\}
\end{aligned}
$$

Substitute into KKT conditions (2.1) - (2.2),

$$
\frac{\mu}{2}(\mathbf{K_X}-\mathbf{B}_1^*-\mathbf{B}_2^*)^{-1}=\boldsymbol{\Psi}_1+\boldsymbol{\Lambda},
$$

$$
\frac{\mu}{2}(\mathbf{K_X}-\mathbf{B}_2^*)^{-1}-\frac{1}{2}(\mathbf{K_Y}-\mathbf{B}_2^*)^{-1}=\boldsymbol{\Psi}_2+\boldsymbol{\Lambda},
$$

The Extremal Inequality

The first term:

- $\mathbf{X}_{2,\gamma} \stackrel{(d)}{=} \check{\mathbf{X}}_{2,\gamma} = \mathbf{X}_{1,\gamma} + \sqrt{\gamma}\check{\mathbf{W}}^G$, where $\check{\mathbf{W}}^G \sim \mathcal{N}(\mathbf{0}, \mathbf{B}_1^*)$.
- By Fisher Information Matrix Inequality and Data Processing Inequality:

$$\begin{aligned}&(\mathbf{K_X} - \mathbf{B}_2^*)J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X} - \mathbf{B}_2^*)\\ \preceq&(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*)J(\mathbf{X}_{1,\gamma}|V)(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*) + \frac{1}{\gamma}\mathbf{B}_1^*\\ \preceq&(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*)J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*) + \frac{1}{\gamma}\mathbf{B}_1^*,\end{aligned}$$

- Therefore,

$$\begin{aligned}&\mathrm{tr}\{[(\mathbf{K_X} - \mathbf{B}_2^*)J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K_X} - \mathbf{B}_2^*) - (\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*)\\ &\quad\cdot J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X} - \mathbf{B}_2^* - \mathbf{B}_1^*) + \mathbf{B}_1^*]\mathbf{\Psi}_1\}\\ \leq&\frac{1+\gamma}{\gamma}\,\mathrm{tr}\{\mathbf{B}_1^*\mathbf{\Psi}_1\} = \mathbf{0}.\end{aligned}$$

The Extremal Inequality

The second term:

- By Cramér–Rao Inequality:

$$J(\mathbf{X}_{2,\gamma}|V)^{-1} \preceq \mathrm{cov}(\mathbf{X}_{2,\gamma}|V) \preceq \mathrm{cov}(\mathbf{X}_{2,\gamma})$$
$$= \mathrm{cov}(\sqrt{1-\gamma}\mathbf{X} + \sqrt{\gamma}\mathbf{X}_2^G) = \mathbf{K}_\mathbf{X} - \gamma \mathbf{B}_\mathbf{2}^*.$$

- Therefore,

$$\mathrm{tr}\{[(\mathbf{K}_\mathbf{X} - \mathbf{B}_\mathbf{2}^*) J(\mathbf{X}_{2,\gamma}|V)(\mathbf{K}_\mathbf{X} - \mathbf{B}_\mathbf{2}^*) - (\mathbf{K}_\mathbf{X} - \mathbf{B}_\mathbf{2}^*)]\mathbf{\Psi}_\mathbf{2}\}$$
$$\geq \mathrm{tr}\{(\mathbf{K}_\mathbf{X} - \mathbf{B}_\mathbf{2}^*)(\mathbf{K}_\mathbf{X} - \gamma\mathbf{B}_\mathbf{2}^*)^{-1}(\gamma - 1)\mathbf{B}_\mathbf{2}^*\mathbf{\Psi}_\mathbf{2}\} = \mathbf{0}.$$

The Extremal Inequality

The third term:

- By Cramér–Rao Inequality:

$$J(\mathbf{X}_{1,\gamma}|U,V)^{-1} \preceq \mathrm{cov}(\mathbf{X}_{1,\gamma}|U,V) \preceq \mathbf{D}.$$

- Therefore,

$$\begin{aligned}&\mathrm{tr}\{[(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})J(\mathbf{X}_{1,\gamma}|U,V)(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\\&\quad -(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})]\mathbf{\Lambda}\}\\&\geq \mathrm{tr}\{(\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*})\mathbf{D}^{-1}[\mathbf{K_X}-\mathbf{B_2^*}-\mathbf{B_1^*}-\mathbf{D}]\mathbf{\Lambda}\}=\mathbf{0}.\end{aligned}$$

$\Rightarrow$ The proof of the extremal inequality is completed.

Rate Region

$$\begin{aligned}
\mu R_1 + R_2 \geq & \mu I(\mathbf{X}; U|V) + I(\mathbf{Y}; V) \\
= & \mu h(\mathbf{X}|V) - \mu h(\mathbf{X}|U,V) - h(\mathbf{Y}|V) + h(\mathbf{Y}) \\
\geq & \frac{\mu}{2} \log \frac{|\mathbf{K_X} - \mathbf{B}_2^*|}{|\mathbf{K_X} - \mathbf{B}_1^* - \mathbf{B}_2^*|} + \frac{1}{2} \log \frac{|\mathbf{K_Y}|}{|\mathbf{K_Y} - \mathbf{B}_2^*|} \\
= & R^*(\mathbf{D}, \mu).
\end{aligned}$$

Thank you!

Q & A ?