Towards Gaussian Capacity, Universality and Short Block Length

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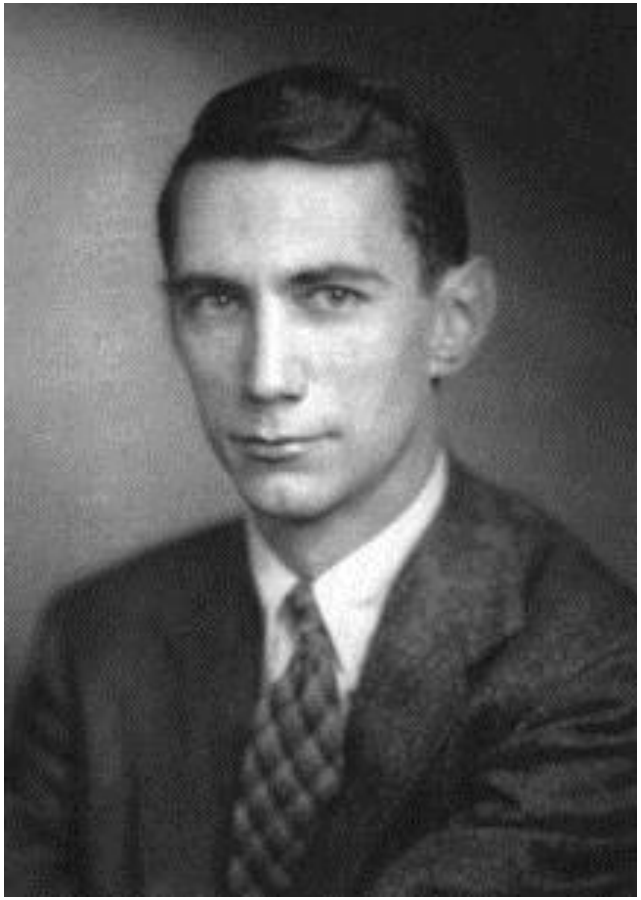
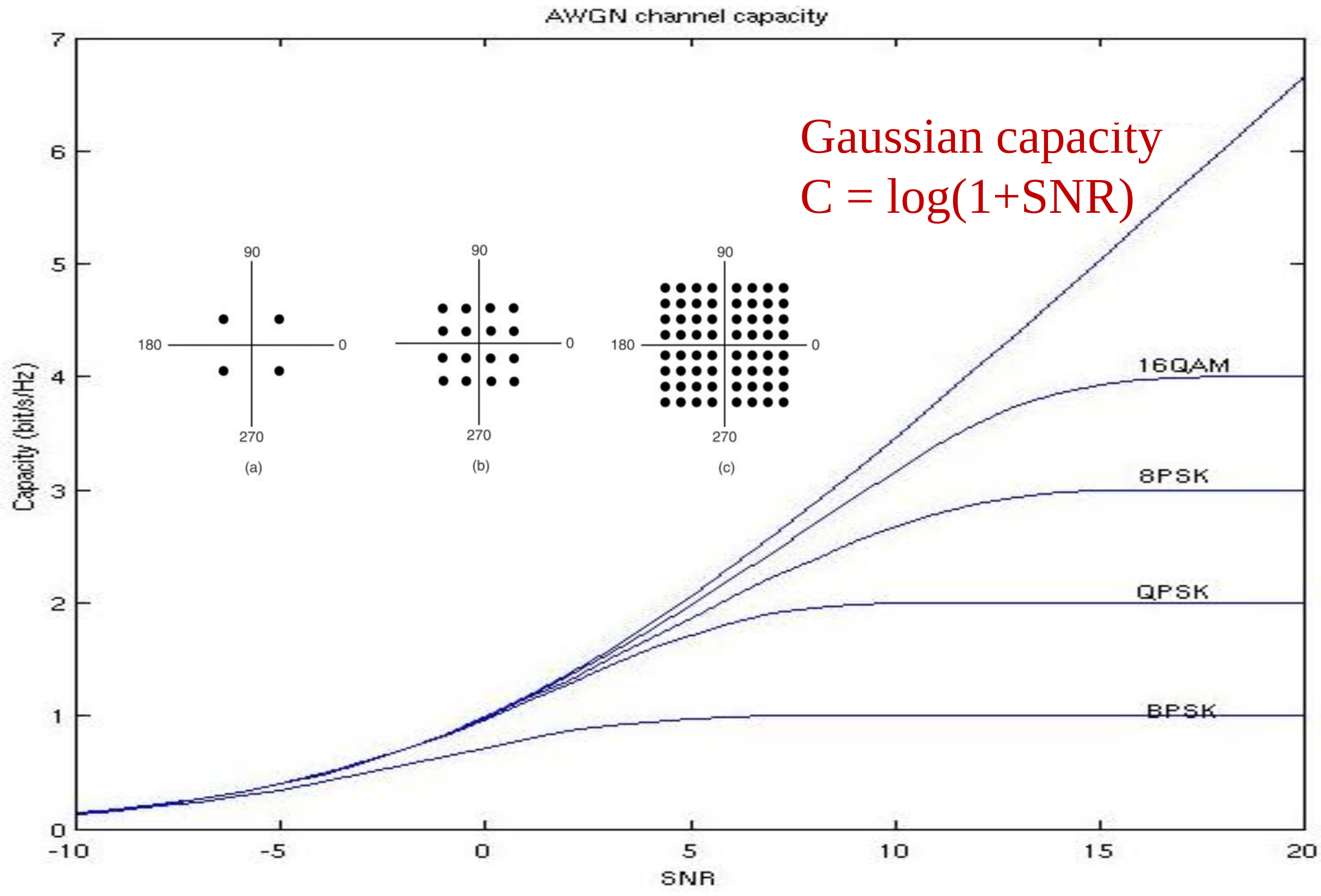
- Background
- Compressed FEC codes
- Spatially-coupled compressed-sensing
- Multi-user systems with short block length per user
- Conclusions

Claude Elwood Shannon 1916 - 2001



100th Birthday Celebration

Shannon capacity



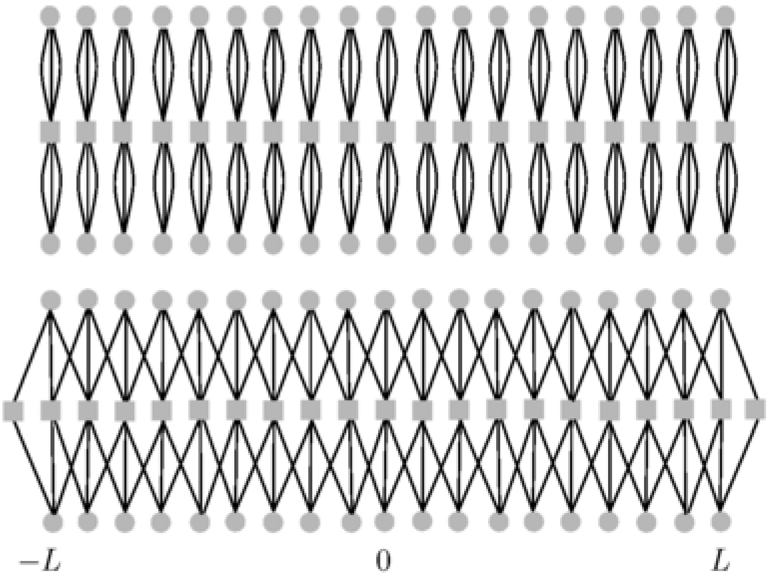
Motivation

Remaining challenges for FEC coding:

- Gaussian capacity,
- universal, for example random puncturing,
- good performance at short block.

Spatially coupled FEC codes

Spatially coupled LDPC codes can offer many advantages, including universality.



- D. G. M. Mitchell, M. Lentmaier, and D. J. Costello, Jr., "Spatially coupled LDPC codes constructed from protographs," *IEEE Trans. Inf. Theory*, vol. 61, no. 9, pp. 4866–4889, Sep. 2015.
- S. Kudekar, T. Richardson, and R. L. Urbanke, "Spatially coupled ensembles universally achieve capacity under belief propagation," *IEEE Trans. Inf. Theory*, vol. 59, no. 12, pp. 7761-7813, Dec. 2013.
- A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, "A simple proof of Maxwell saturation for coupled scalar recursions," *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 6943-6965, Nov. 2014.

Spatially coupled FEC codes

We are also inspired by the following work:

- K. Takeuchi, T. Tanaka, and T. Kawabata, "Improvement of BP-based CDMA multiuser detection by spatial coupling," in *Proc. IEEE Int. Symp. Inf. Theory*, Russian, Jul.-Aug. 2011, pp. 1489-1493.
- D. Truhachev and C. Schlegel, "Spatially coupled streaming modulation," in *Proc. IEEE Int. Conf. Commun.*, Hungary, Jun. 2013, pp. 3418-3422.
- A. Yedla, P. S. Nguyen, H. D. Pfister, and K. R. Narayanan, "Universal codes for the Gaussian MAC via spatial coupling," in *Proc. Allerton Conf. Commun., Contr. & Comput.*, USA, Sep. 2011, pp. 1801-1808.
- A. Joseph and A. R. Barron, "Fast sparse superposition codes have near exponential error probability for $R < C$," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 919-942, Feb. 2014.

Sparse regression codes

In an sparse regression (SR) code, information is encoded in the sparsity of the transmitted signal.

Sparsity implies low rate. Compressed sensing is used here to increase rate.

AMP is used for decoding.

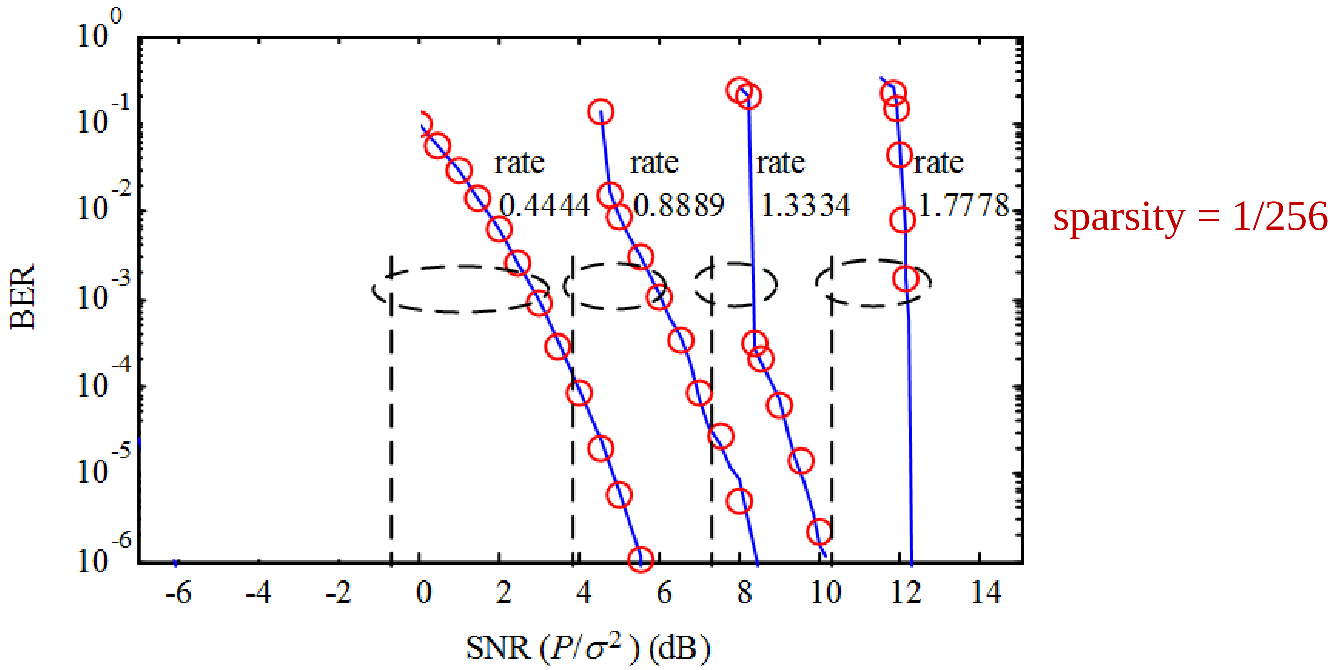
$$\begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0.52 & 0.67 & 0.65 & 0.53 & -2.30 & -0.89 & 0.85 & -0.11 \\ 0.21 & -0.84 & -1.62 & -2.06 & -1.16 & -0.02 & 0.21 & 0.93 \\ -1.34 & -1.38 & -0.37 & -0.22 & 2.09 & -0.62 & 0.93 & -1.40 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \text{noise}$$

- A. Joseph and A. R. Barron, "Fast sparse superposition codes have near exponential error probability for $R < C$," *IEEE Trans. Inf. Theory*, vol. 60, no. 2, pp. 919-942, Feb. 2014.
- Davis L. Donoho, Arian Maleki, and Andrea Montanari. "Message-passing algorithms for compressed sensing." in *Proceedings of the National Academy of Sciences*, 2009.

Sparse regression (SR) codes

SR codes can achieve capacity asymptotically. However, they suffer from the following difficulties.

- A SR code can achieve capacity only at very high sparsity.
- High sparsity implies high decoding complexity.
- With limited sparsity, SR codes are good only at very high rates.



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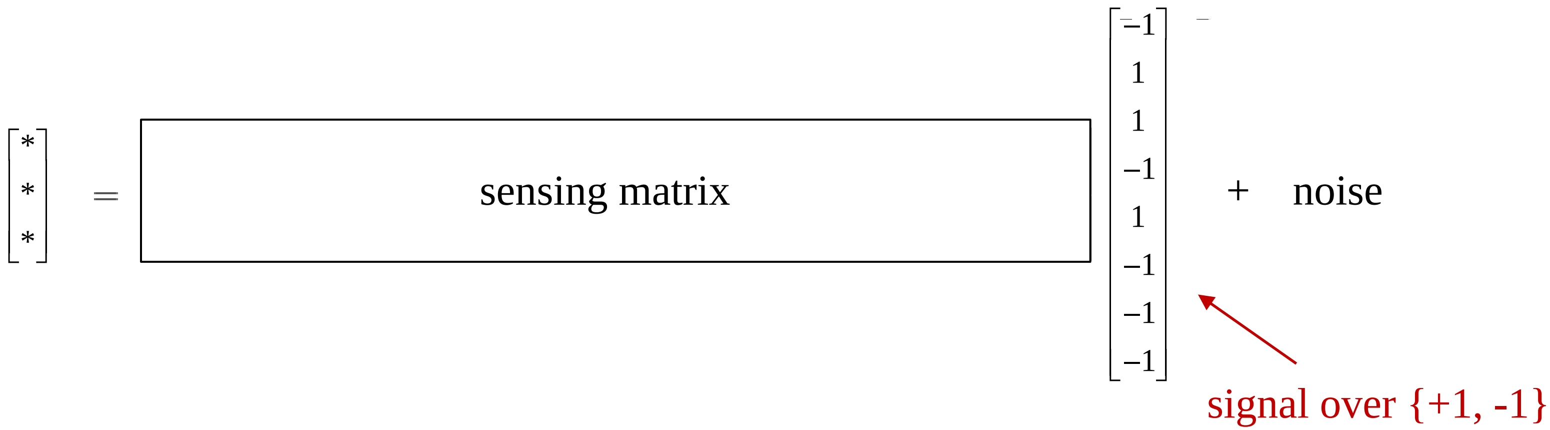
Compression of non-sparse signals

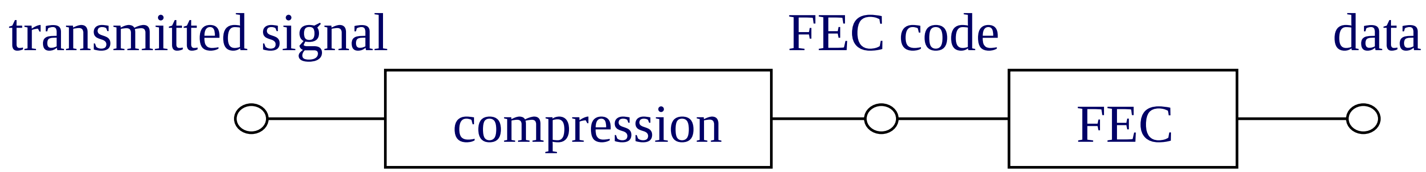
It is usually assumed that the signal to be compressed is sparse. This is actually not necessary.

The signal can be drawn from a general constellation. AMP works well in such general situations.



Junjie Ma and Li Ping, "Orthogonal AMP for Compressed Sensing with Unitarily-invariant Matrices," IEEE ITW, September 2016, Cambridge UK.

Compressed FEC coding



a FEC codeword

$$
\begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0.52 & 0.67 & 0.65 & 0.53 & -2.30 & -0.89 & 0.85 & -0.11 \\ 0.21 & -0.84 & -1.62 & -2.06 & -1.16 & -0.02 & 0.21 & 0.93 \\ -1.34 & -1.38 & -0.37 & -0.22 & 2.09 & -0.62 & 0.93 & -1.40 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \text{noise}
$$

Note the opposite effects of FEC coding and compression:

- FEC coding introduces redundancy.
- Compression reduces redundancy.
- Rate can be easily adjusted in this way.

Hadamard sensing matrix

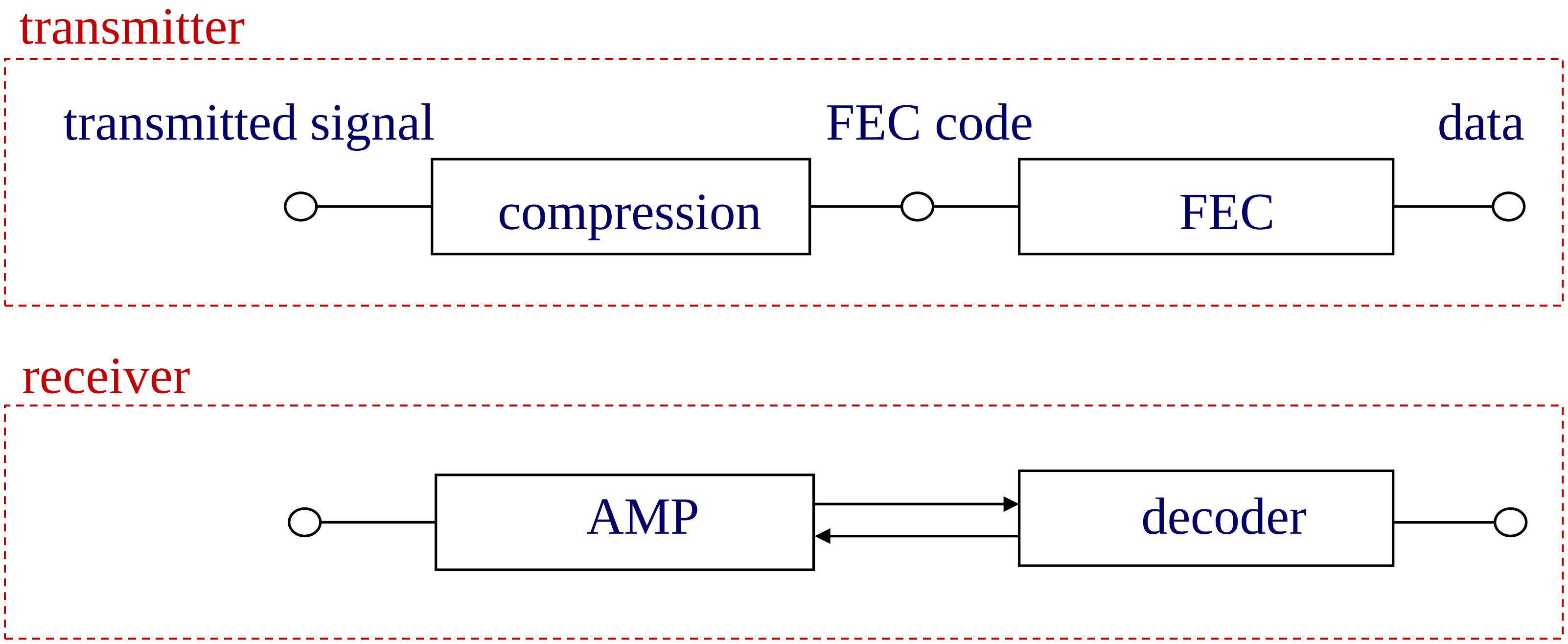
Complexity is high in handling a random compression matrix. This difficult can be alleviated by adopting a Hadamard sensing matrix. The rows of a Hadamard matrix are orthogonal. Fast Hadamard transform (FHT) can be applied for efficient detection.

$$\begin{bmatrix} * \\ * \\ * \end{bmatrix} = \boxed{\text{randomly selected rows from a Hadamard matrix}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \text{noise}$$

Junjie Ma and Li Ping, "Orthogonal AMP for Compressed Sensing with Unitarily-invariant Matrices," IEEE ITW, September 2016, Cambridge UK.

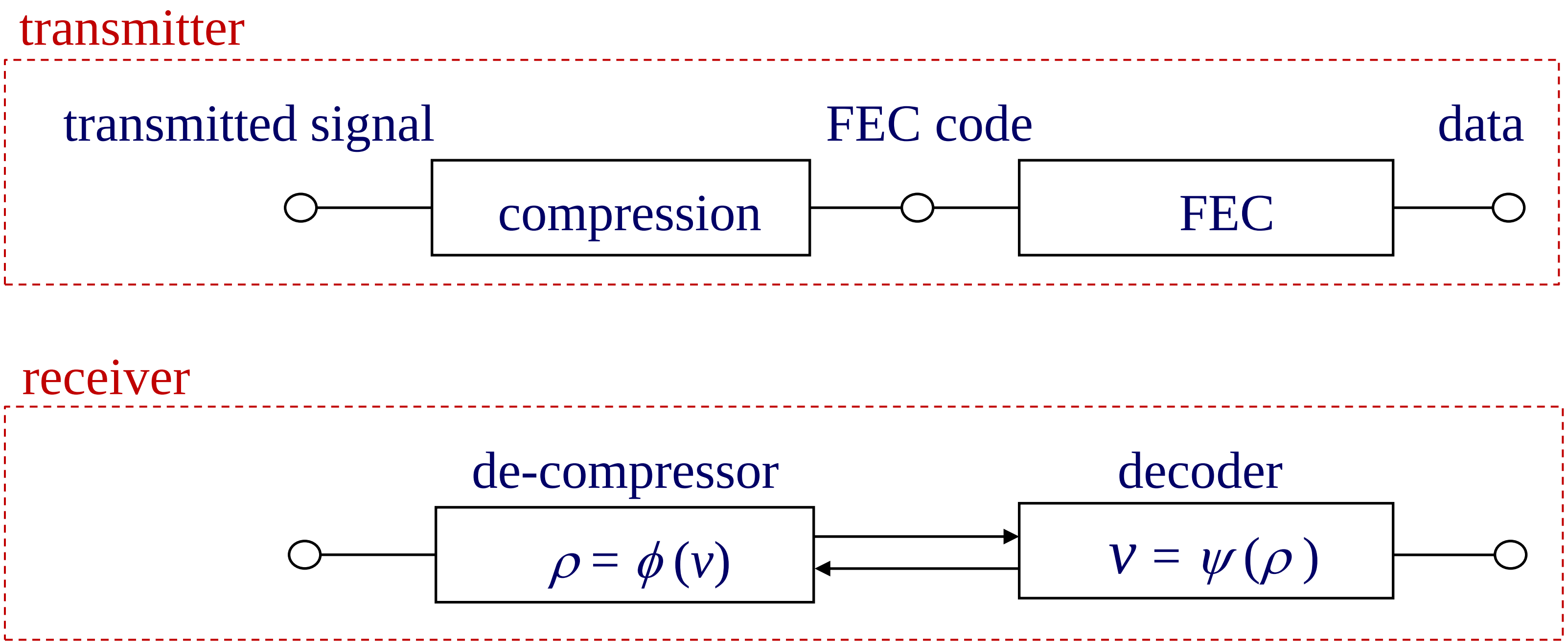
Iterative receiver

Detection can be carried in an iterative way.



Evolution analysis

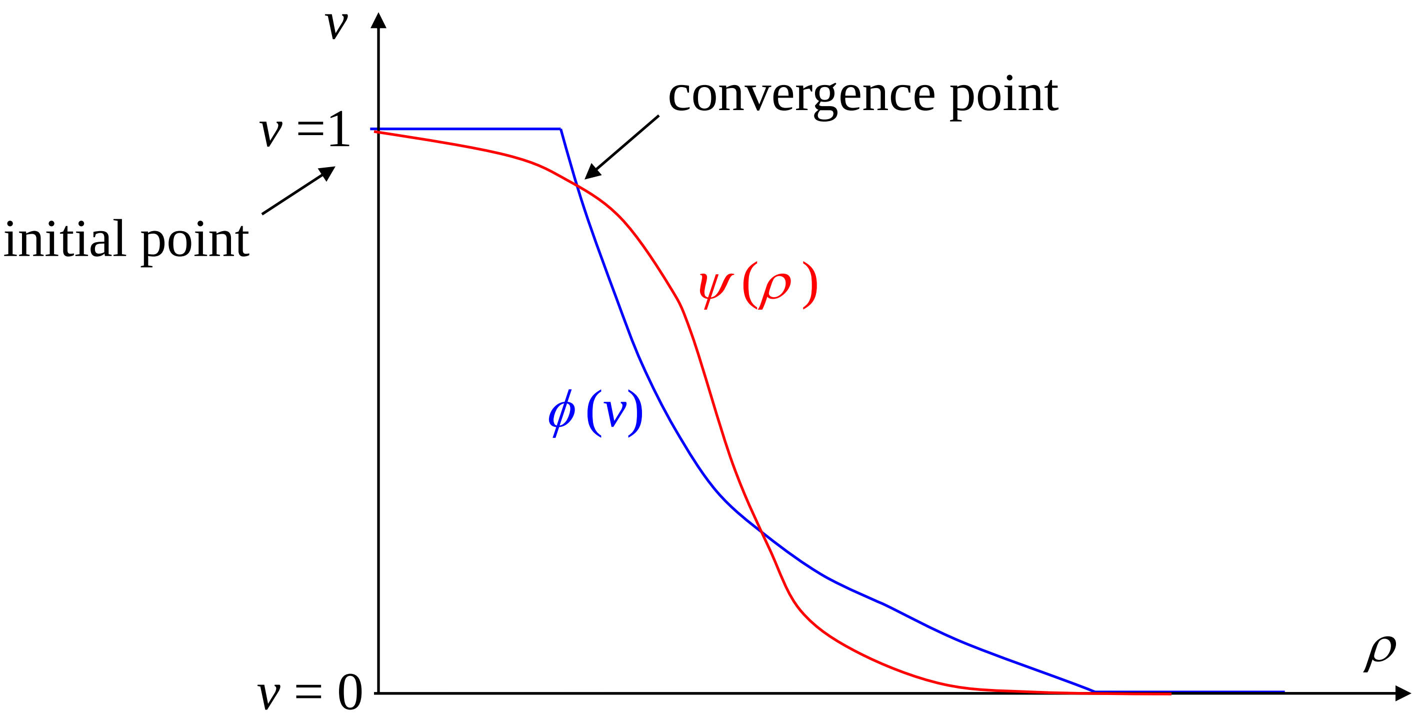
Performance can be assessed using evolution analysis.



- M. Bayati and A. Montanari "The dynamics of message passing on dense graphs, with applications to compressed sensing," in *IEEE Trans. Inf. Theory*, Feb. 2011.
- Junjie Ma and Li Ping, "Orthogonal AMP for Compressed Sensing with Unitarily-invariant Matrices," IEEE ITW, September 2016, Cambridge UK.

Poor convergence

Compressed FEC coding with iterative decoding has poor convergence.



S. ten Brink, “Convergence behavior of iteratively decoded parallel concatenated codes”, *IEEE Trans. Commun.*, vol. 49, no. 10, Oct. 2001.

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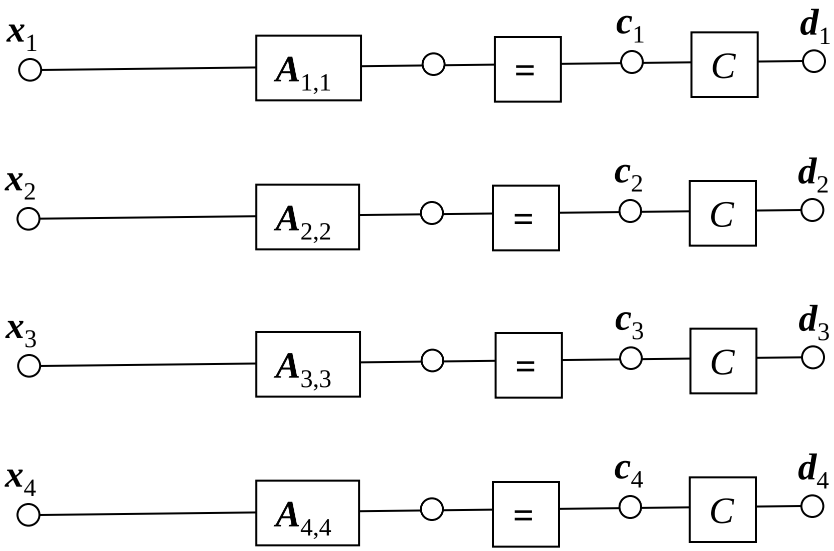
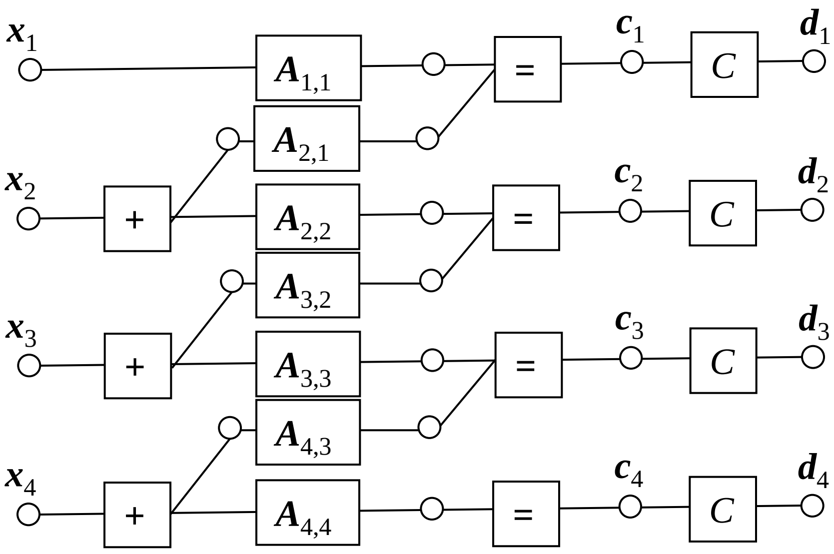
Start from a compressed FEC coding system

Four separate compressed FEC coding systems.



Spatially-coupled compressed-sensing

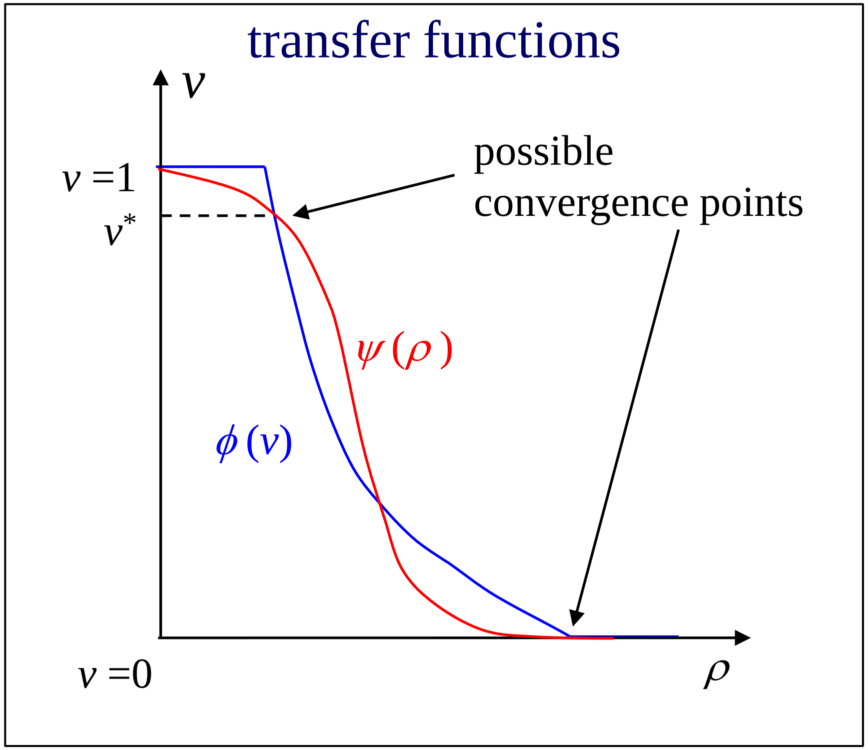
A SCCS system with coupling width W=2.

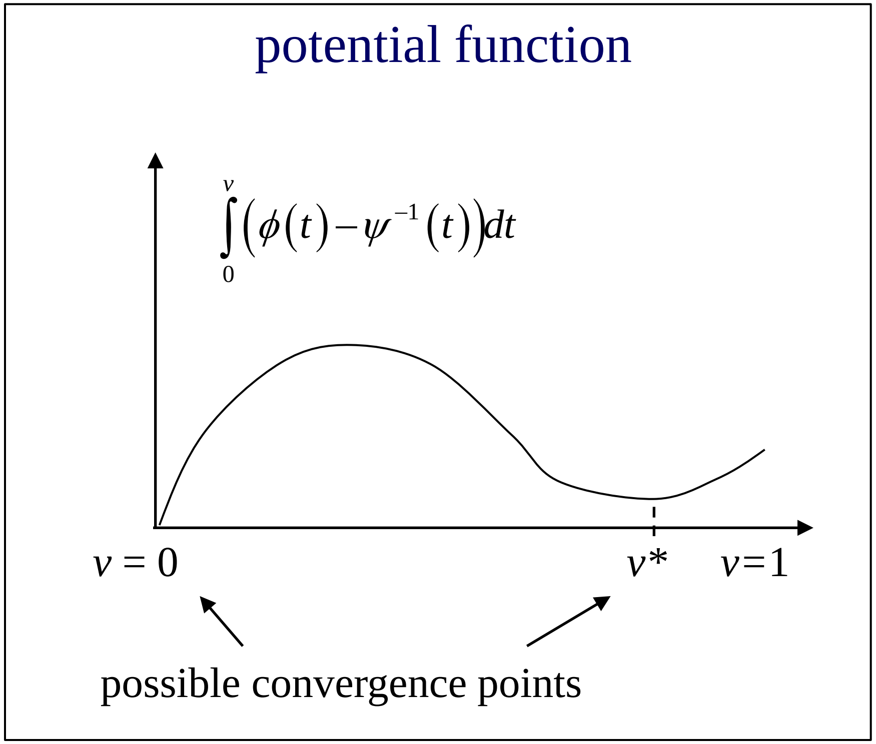


Potential function analysis

It is shown in the following paper that the iterative process converges to v = 0 provided that the potential function is above zero for all v>0.

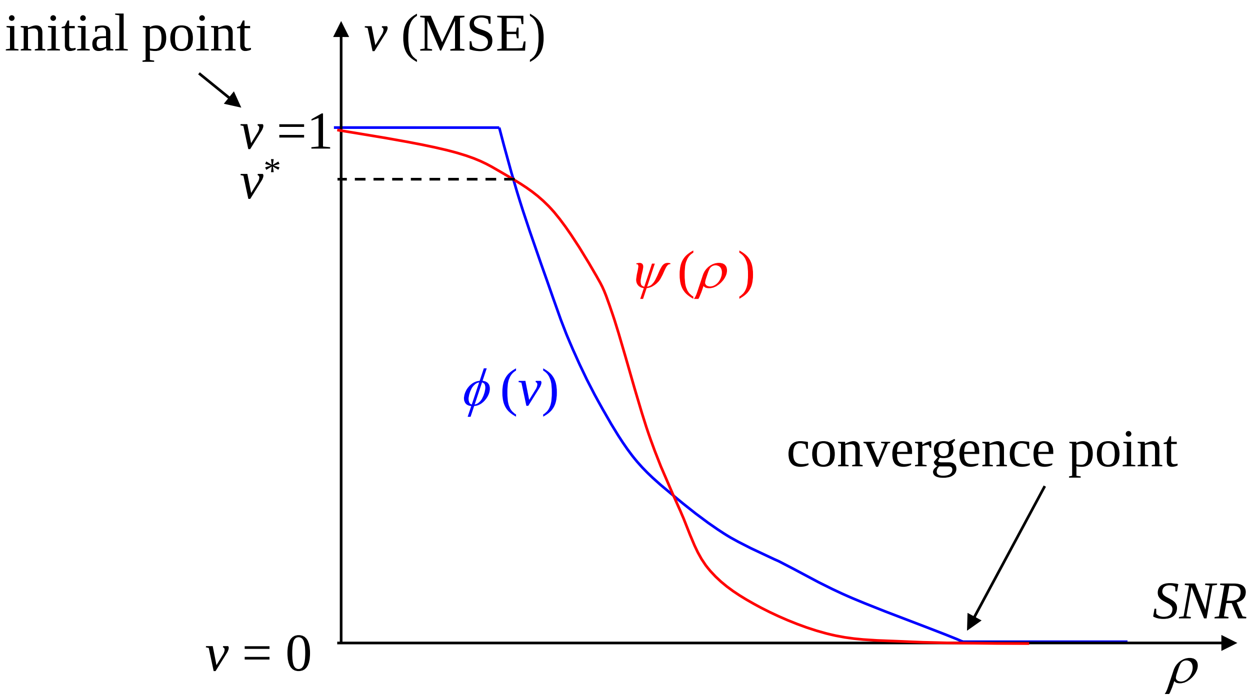
A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, "A simple proof of Maxwell saturation for coupled scalar recursions," IEEE Trans. Inf. Theory, Nov. 2014.





Convergence condition

Therefore the iterative process converges to $v = 0$ provided that the area covered by $\phi(v) - \psi^{-1}(v)$ is always positive.



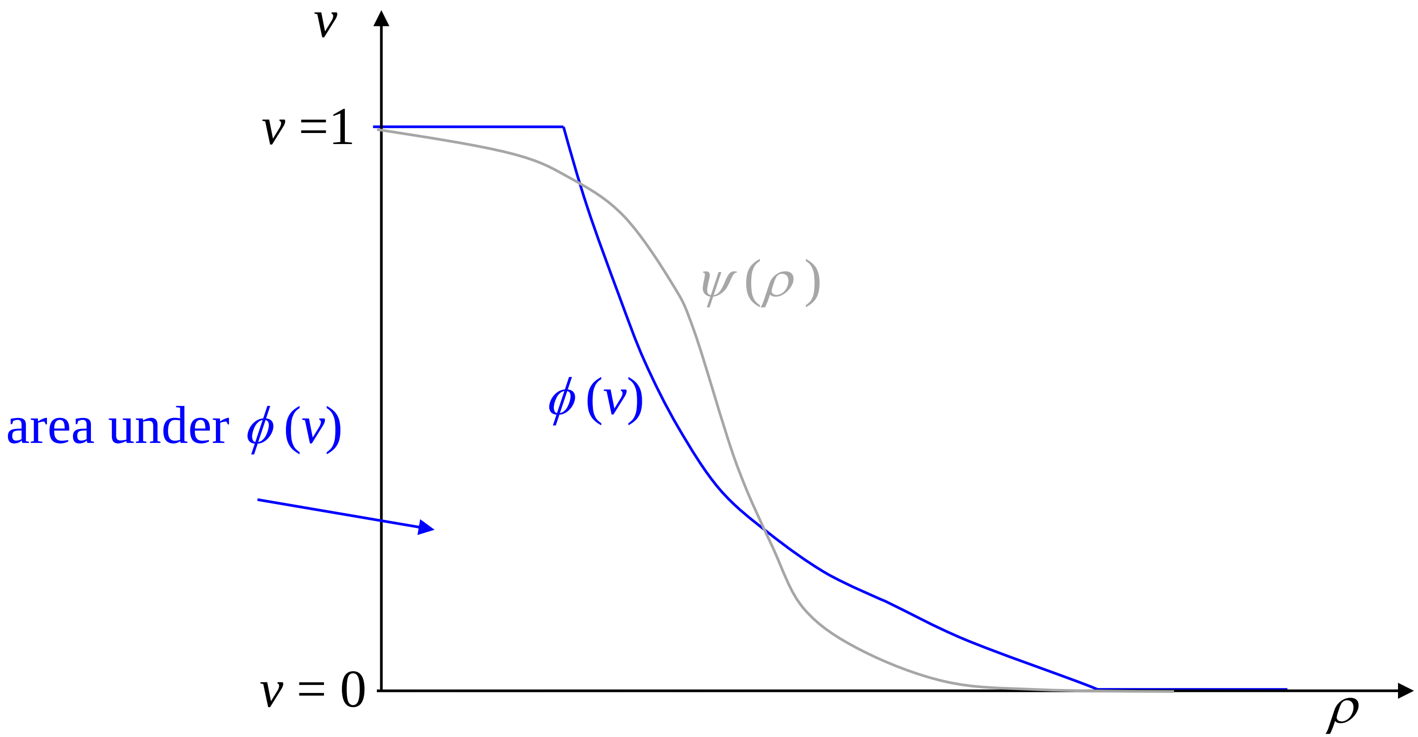
Main result

Theorem 1: A spatial-coupled and compressed FEC coding system can approach Gaussian channel capacity at any rate under iterative decoding.

Proof of Theorem 1

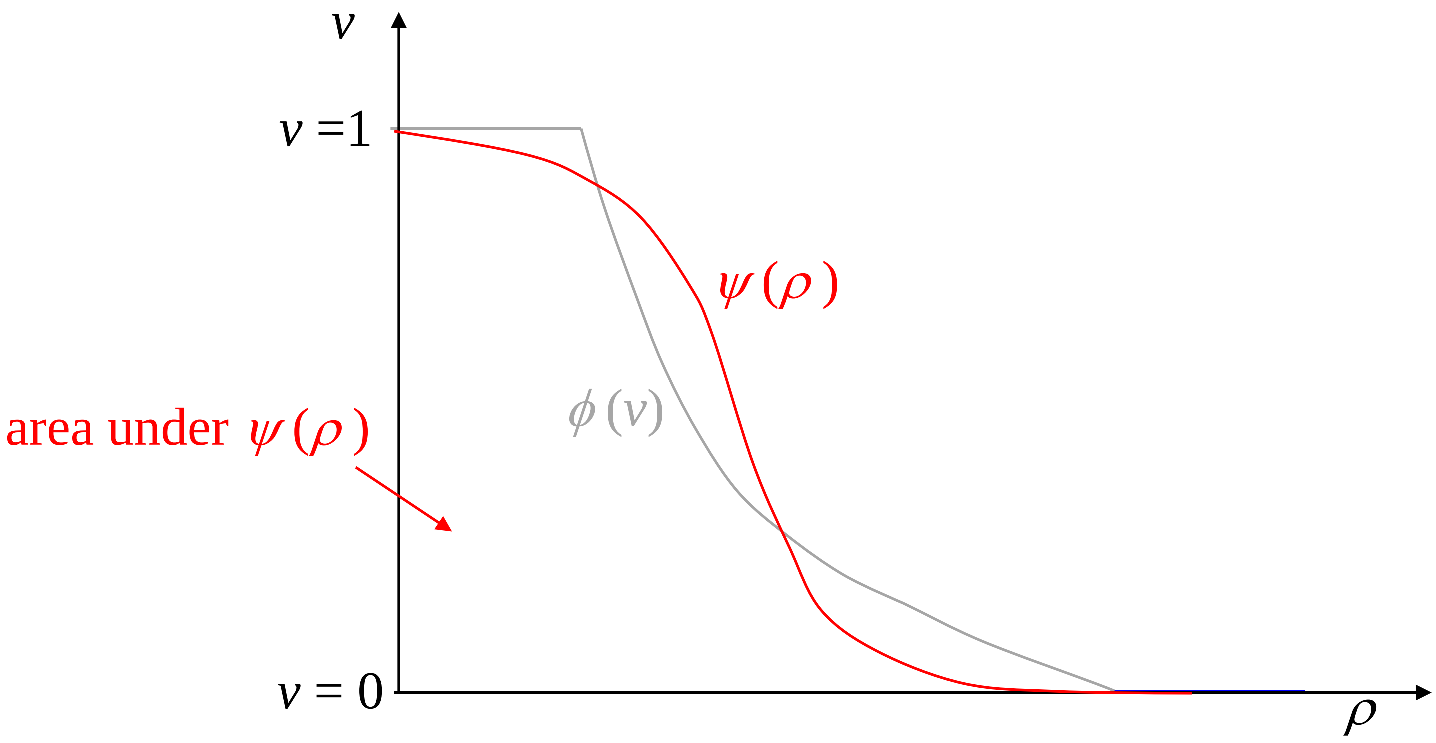
Lemma 1: The area under $\rho = \phi(v)$ equals the channel capacity C_{CH} scaled by the compression ratio:

$$\frac{1}{2}\int_0^1 \phi(v)\,\mathrm{d}v = \frac{M}{N}\cdot C_{\mathrm{CH}}$$



Proof of Theorem 1 (continued)

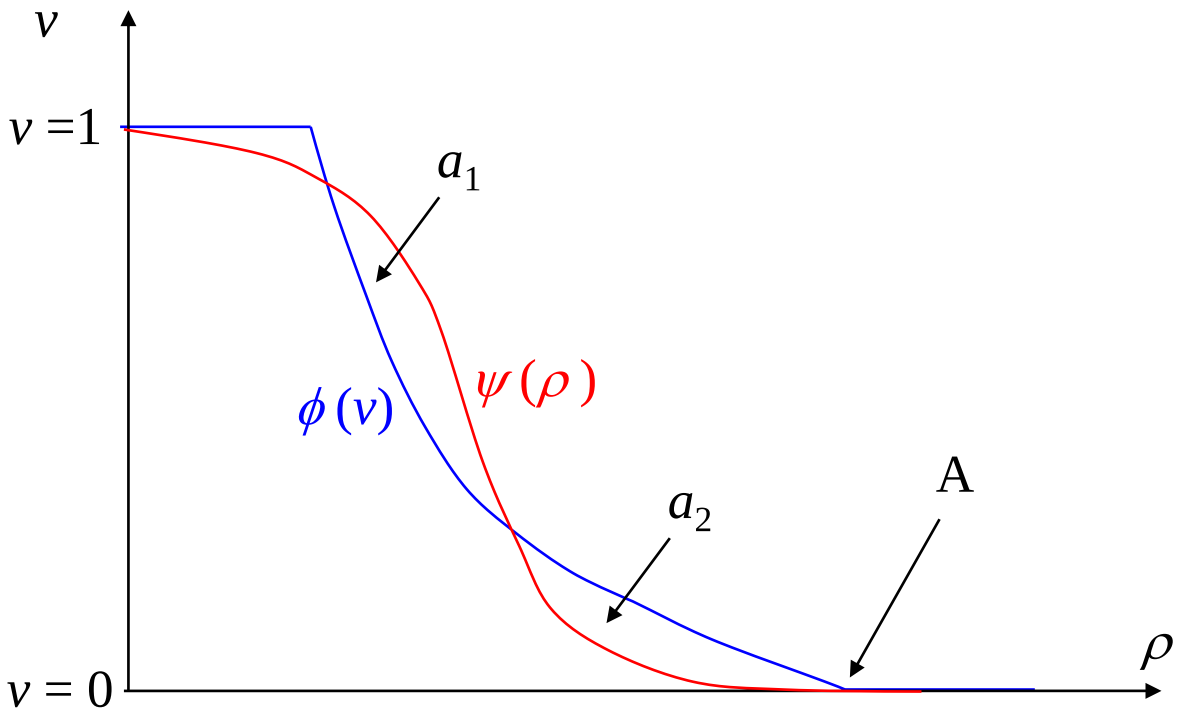
Lemma 2: The area under $v = \psi(\rho)$ equals the rate R of the FEC code.



K. Bhattad and K. R. Narayanan, "An MSE-based transfer chart for analyzing iterative decoding schemes using a Gaussian approximation," *IEEE Trans. Inf. Theory*, Jan. 2007.

Proof of Theorem 1 (continued)

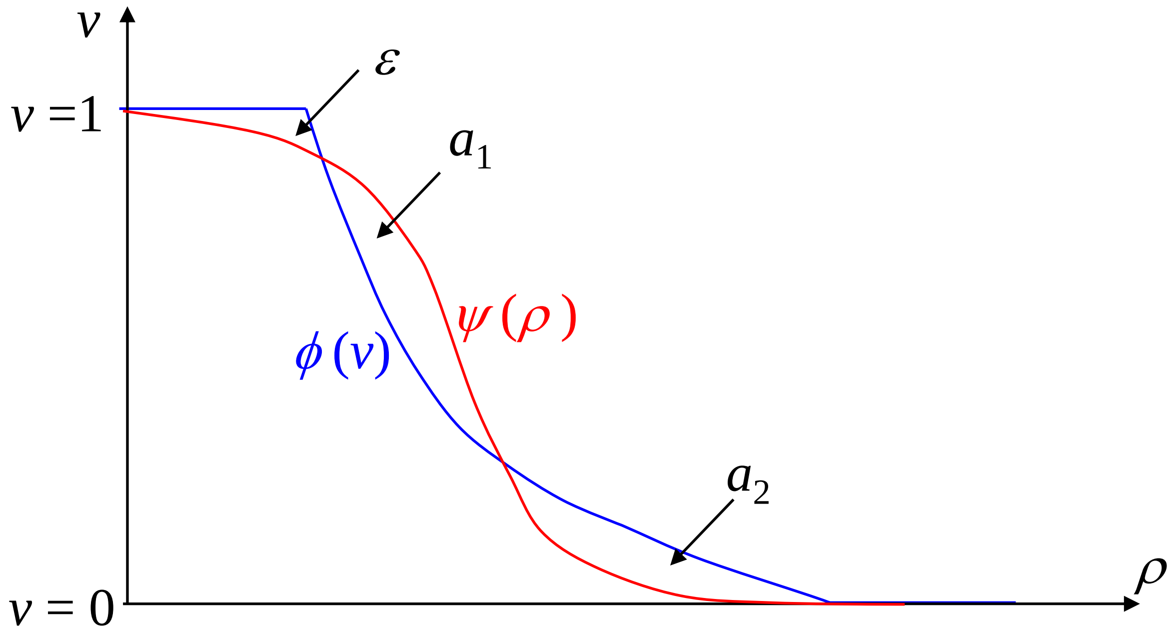
Lemma 3: With spatial coupling and provided that $a_1 < a_2$, iterative decoding converges to A.



A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, "A simple proof of Maxwell saturation for coupled scalar recursions," IEEE Trans. Inf. Theory, Nov. 2014.

Proof of Theorem 1 (continued)

Lemma 4: When $a_1 = a_2$, the areas under $\phi(v)$ and $\psi(\rho)$ differs by only $\varepsilon N/M$, where M/N is the compression ratio.

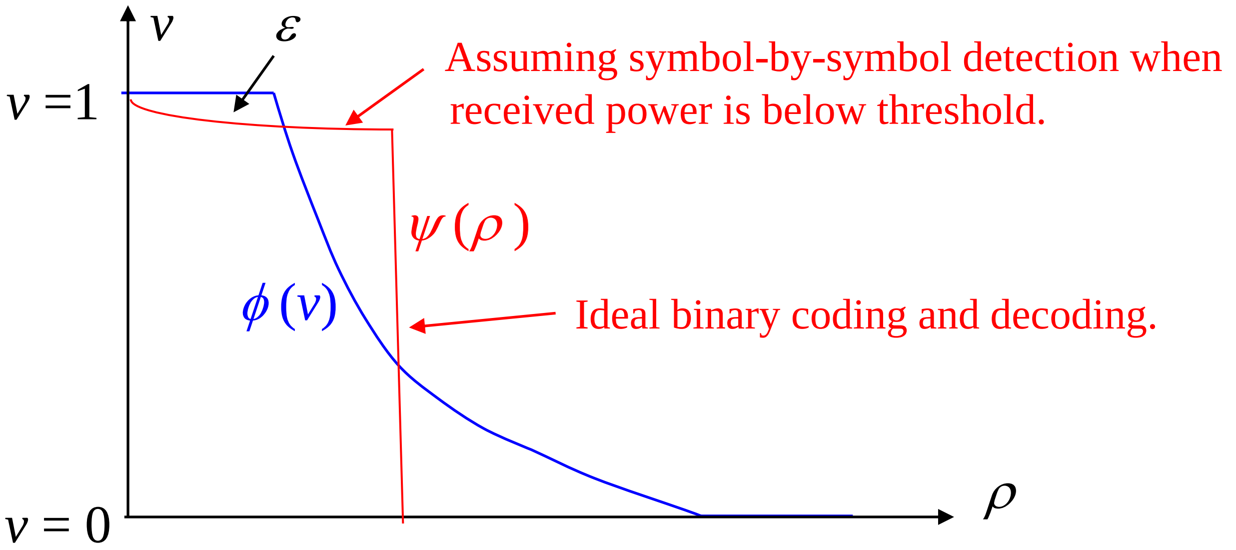


Proof of Theorem 1 (continued)

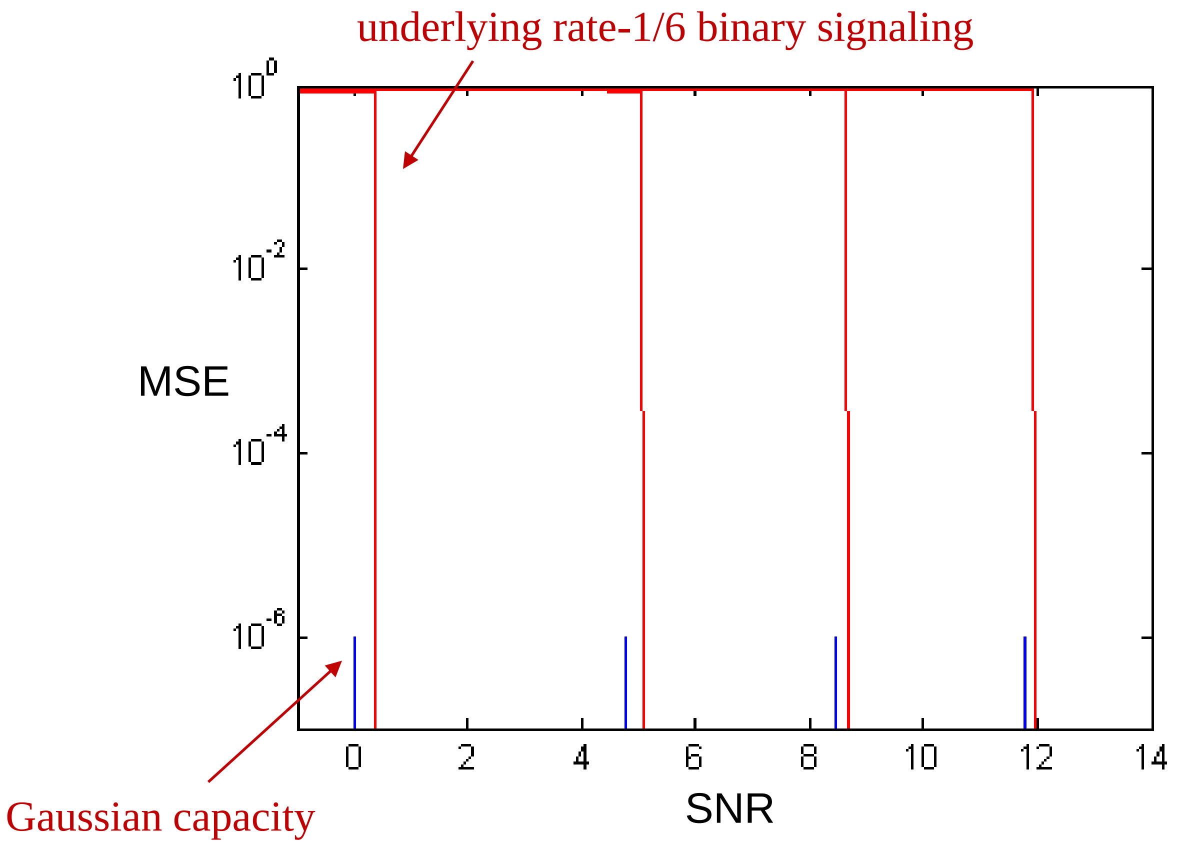
Finally, we can show that $\varepsilon N/M \to 0$ if the underlying binary FEC code achieves capacity and its rate is sufficiently low. Then the areas under $\phi(v)$ and $\psi(\rho)$ are equal, i.e., (from Lemmas 1 and 2),

$$R_{\text{CS}} = C_{\text{CH}} + o\left(\frac{M}{N}\right)$$

where $R_{\text{CS}} = (M/N)R$ is the coding rate after compression. This completes the proof.



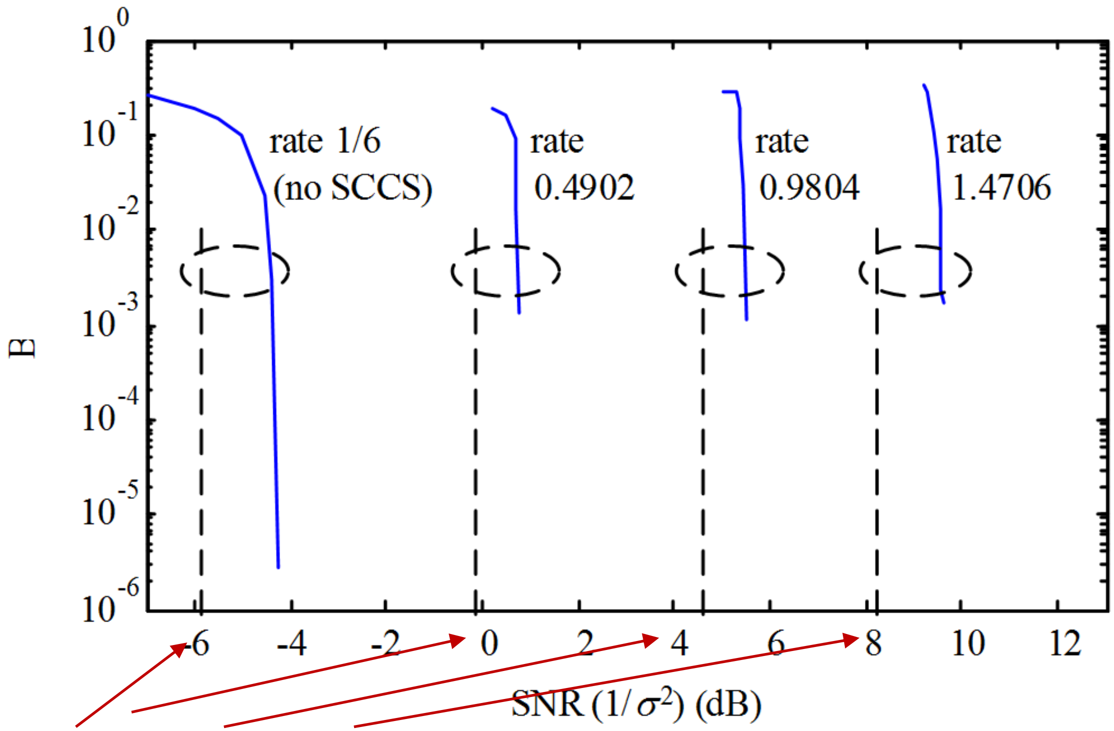
Gaussian capacity and universality-evolution results



Analysis shows that the compressed FEC coding scheme can operates very close to Gaussian capacity universally, even with binary underlying signaling.

Gaussian capacity and universality-simulation results

Properly designed binary FEC codes with random compression can significantly outperform SR codes.



Gaussian capacities

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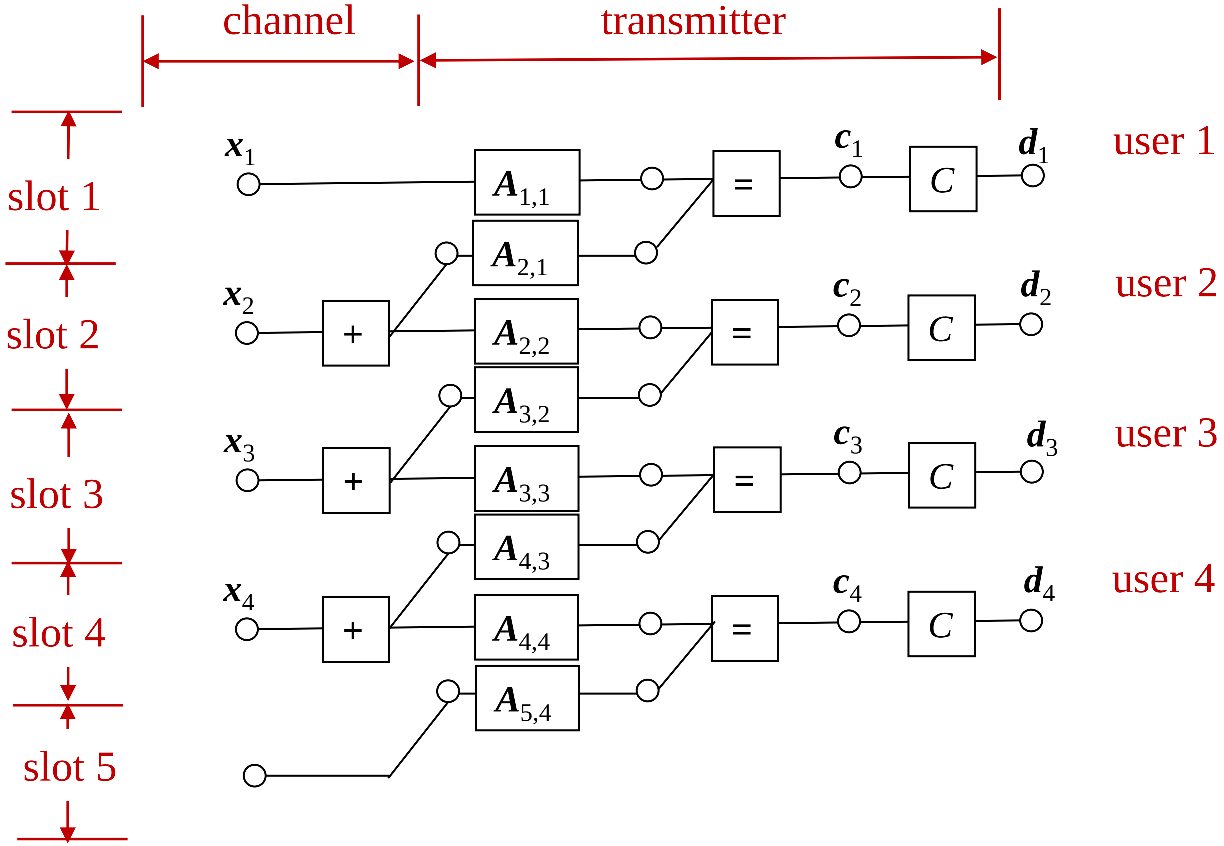
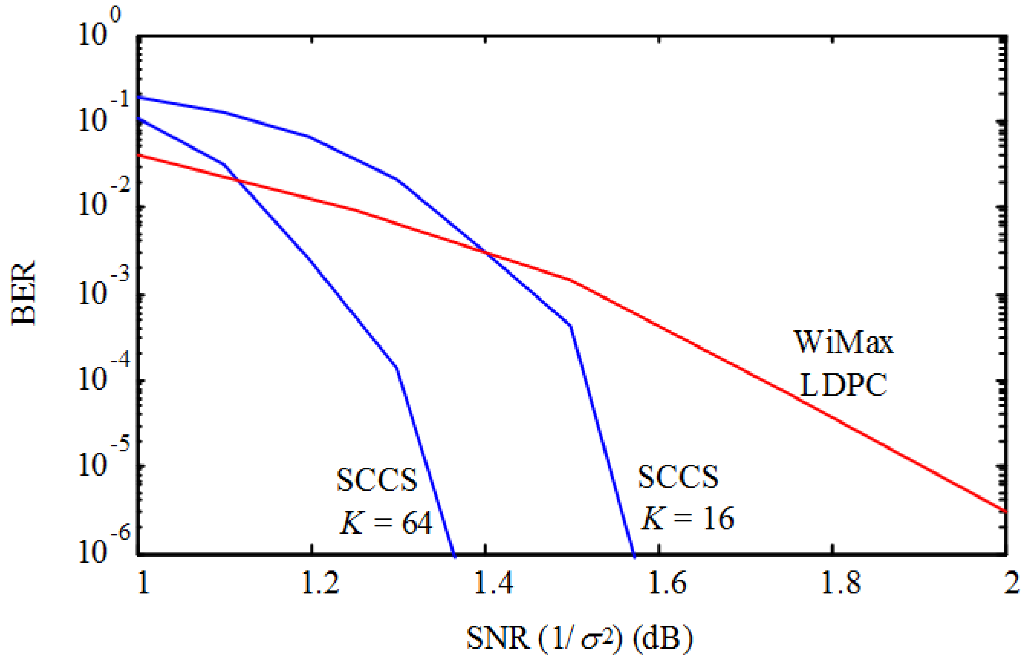
SCCS in a multi-user system

A multi-user SCCS system with coupling width W=2. Users are separated using different interleavers, which follows the IDMA principle.



Short coding length

Good short codes can be designed for IDMA type multi-user systems. The following is an example.



For all codes, rate ≈ 0.5 and length ≈ 1024.

The block length problem

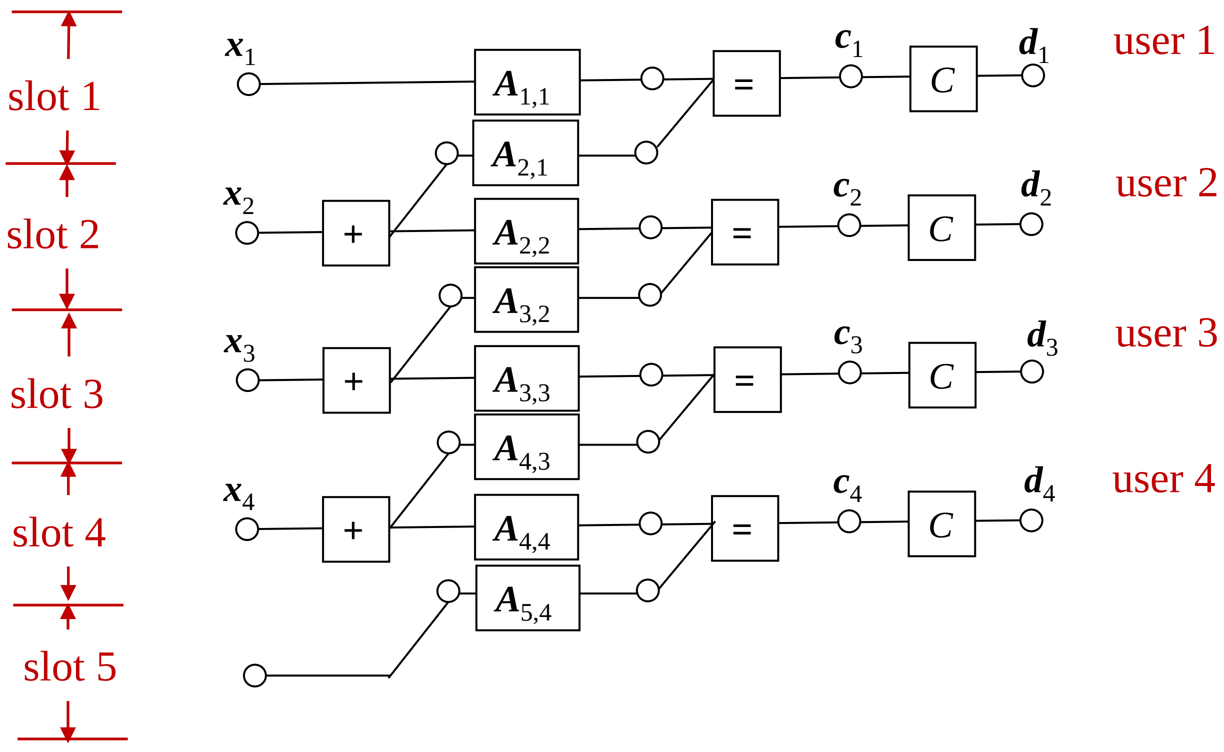
The block length problem is usually due to the latency constraint. It cannot be solved by, e.g., increasing receiver speed. It is a source problem; the source can only generate a limited number of information bits within a fixed duration in many real-time applications.

Multi-user SCCS still requires long total length. Nevertheless, the block length for each user is short.

Intuition

Effectively, each user transmits in more time slots. This results in lower rate and higher coding gain, provided that cross user interference can be ignored.

However, interference does exist in SCCS. It appears that SCCS provides an efficient way for multiuser interference cancelation.



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Conclusions

SCCS can potentially approaches Gaussian capacity.

SCCS is universally good for a quite wide range of puncturing ratio.

In a multiuser system, SCCS can be implemented with low latency per user.