# Transform Technique and Interference Alignment for Acyclic Delay Networks

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# Outline of Presentation I

- Underdetermined (?) systems
- Applications of IA
  - Gaussian Interference Channel
  - Three Unicast Sessions with Linear Network Coding
- Transform Technique for Acyclic Networks with Delay
- IA for 3-Unicast Sessions with Delay
  - IA with Time-Invariant LECs
  - IA with Block Time-Varying LECs
  - IA with Time-Varying LECs
- Conclusion

# Outline

# Many Variables, Few Equations [Jaf]

- Consider linear system in several variables.
- We have following system of equations with $M < N$:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_M \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ H_{M1} & H_{M2} & \cdots & H_{MN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

- Interested only in $X_1$. Rest are interference.

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[Jaf] Prof. Syed Jafar's monograph freely available at: https://sites.google.com/site/interferencealignment/

# Many Variables, Few Equations (Contd.)

- A realization $x_1$ can be decoded iff

$$\begin{bmatrix} H_{11} \\ H_{21} \\ \vdots \\ H_{M1} \end{bmatrix} \notin Span \left\{ \begin{bmatrix} H_{12} \\ H_{22} \\ \vdots \\ H_{M2} \end{bmatrix}, \cdots, \begin{bmatrix} H_{1N} \\ H_{2N} \\ \vdots \\ H_{MN} \end{bmatrix} \right\}$$

  - If $\{\text{Signal Space} \cap \text{Interference Space}\} \neq \{\phi\}$ then, contribution of $x_1$ to the same received vector will be non-unique.

# Many Variables, Few Equations (Contd.)

Example 1: $\begin{bmatrix} H_{11} \\ H_{21} \\ \vdots \\ H_{M1} \end{bmatrix} \in Span \left\{ \begin{bmatrix} H_{12} \\ H_{22} \\ \vdots \\ H_{M2} \end{bmatrix}, \cdots, \begin{bmatrix} H_{1N} \\ H_{2N} \\ \vdots \\ H_{MN} \end{bmatrix} \right\}$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} X_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} X_3,$$

for $\{X_1, X_2, X_3\} = \{0, 1, -1\}$ and $\{X_1, X_2, X_3\} = \{1, 0, 0\}$.

# Example: Many Variables, Few Equations (Contd.)

- Consider the following system of equations, over $\mathbb{C}$.

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}}_{\underline{h_1}} X_1 + \underbrace{\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}}_{\underline{h_2}} X_2 + \underbrace{\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}}_{\underline{h_3}} X_3 + \underbrace{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}}_{\underline{h_4}} X_4 + \underbrace{\begin{bmatrix} 5 \\ 5 \\ 8 \end{bmatrix}}_{\underline{h_5}} X_5$$

- $\underline{h_4} = \underline{h_3} - \underline{h_2}$, $\underline{h_5} = \underline{h_3} + \underline{h_2}$
- Interference vectors $\underline{h_2}$ and $\underline{h_3}$ span 2-D space.
- $[17 \; -1 \; -10]$ is orthogonal to $\underline{h_2}$ and $\underline{h_3}$, but not to $\underline{h_1}$.

# Objective

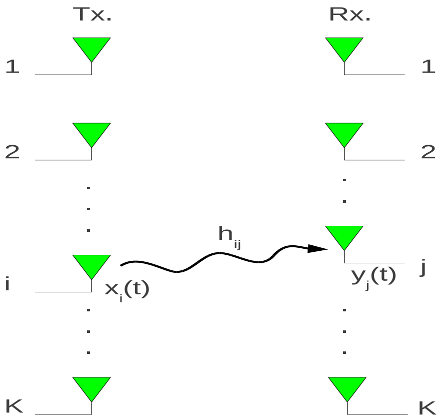
Constraining interference symbols in a sub-space that is linearly independent of the signal sub-space over some field $\mathbb{F}$.

- Challenges:
  - Simultaneously achieving IA at several receivers.

# Outline

# IA in 3-User Gaussian Interference Channel

# $K$-User Gaussian Interference Channel



# Sum Degrees of Freedom

## Definition

A Gaussian Interference Network is said to have a sum degrees of freedom (DoF) of $d$, if its sum-capacity scales as

$$C_{sum} = d \log SNR + o(\log SNR)$$

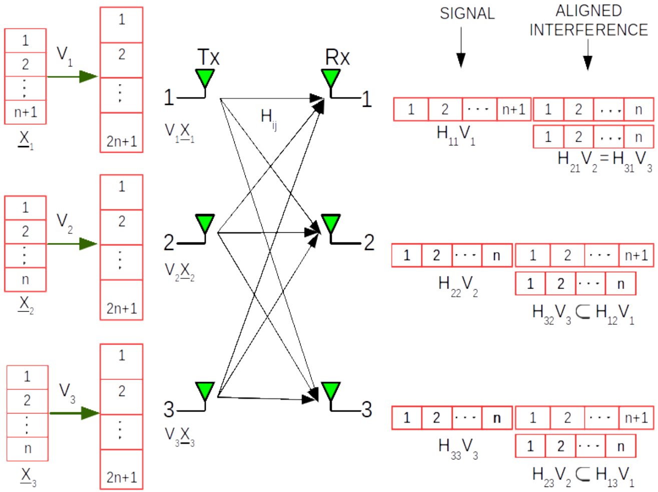
- Sum-DoF of $K$-user GIC is upper-bounded by $\frac{K}{2}$.

# Asympotic IA in 3-User GIC with Random Channel Coefficients [CaJ]

- Sum-DoF($K$-User GIC)$= K/2$.
- Objective: Achieve a sum-DoF of $\frac{3}{2} - \epsilon$ using IA in 3-User GIC, for some $\epsilon > 0$.
- Create $2n$-equations, at each receiver, in $3n$-variables, with $n$ variables assigned to each Tx.

[CaJ] V. Cadambe, S. Jafar, "Interference Alignment and Degrees of Freedom for the $K$ user Interference Channel", IEEE Trans. Info. Theory, Vol. 54, no. 8, pp. 3425-3441, Aug. 2008.

# Asympotic IA in 3-User GIC (Contd.)



# Asympotic IA in 3-User GIC (Contd.)

- Input-output relation is given by

$$Y_j = \sum_{i=1}^{3} H_{ij} V_i \underline{X_i} + N_j.$$

| | | | |
|---|---|---|---|
| Precoders | $V_1 \rightarrow 2n+1 \times n+1$ | $V_2 \rightarrow 2n+1 \times n$ | $V_3 \rightarrow 2n+1 \times n$ |
| Data Symbols | $\underline{X_1} \rightarrow n+1 \times 1$ | $\underline{X_2} \rightarrow n \times 1$ | $\underline{X_3} \rightarrow n \times 1$ |

# Asympotic IA in 3-User GIC (Contd.)

- Structure of $H_{ij}$: $H_{ij} = \begin{bmatrix} h_{ij}^{(1)} & 0 & \cdots & 0 \\ 0 & h_{ij}^{(2)} & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & h_{ij}^{(2n+1)} \end{bmatrix}$

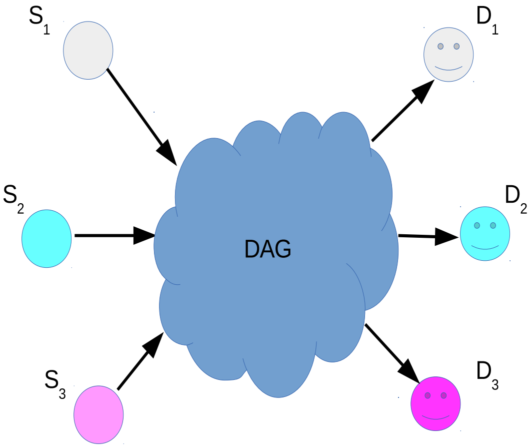
# Selection of IA Precoders in 3-User GIC

- $V_1 = [W \;\; TW \;\; \cdots \;\; T^n W]$.
- $V_2 = H_{23}^{-1} H_{13} V_1 A$.
- $V_3 = H_{23}^{-1} H_{13} V_1 B$
  - $T = H_{12}^{-1} H_{32} H_{31}^{-1} H_{21} H_{23}^{-1} H_{13}$
  - $A$ : Selection matrix - selects first $n$ columns
  - $B$ : Selection matrix - selects last $n$ columns

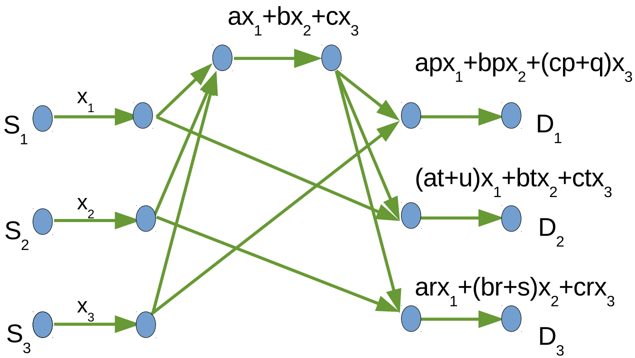
RESULT: The above choice of precoders achieves IA almost surely and DoFs of $\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1}$ are achieved. A sum-DoF of $\frac{3}{2}$ achieved asymptotically.

# IA in Three Unicast Sessions with Linear Network Coding

# Three Unicast Sessions



# Why IA for 3-Unicast Sessions?




Impossible to achieve zero-interference!

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[MDRJMV] C. Meng et al., "Precoding-Based Network Alignment For Three Unicast Sessions", arXiv:1305.0868 [cs.IT].

# Why IA for 3-Unicast Sessions (Contd.)

- Systematic code design that guarantees rate of $\frac{1}{2}$ (asymptotically) for every source-destination pair for a wide range of network topologies.
- Random linear coding at intermediate nodes.

# Assumptions

- Mincut($S_i - D_i$)=1, Mincut($S_i - D_j, j \neq i$)$\geq$1.
- Link capacity $=$ one $\mathbb{F}_{2^m}$ symbol per link use.
- Input-output relation at $D_j$ is given by [KoM]

$$Y_j = \sum_{i=1}^{3} M_{ij}(\underline{\epsilon}) X_i$$

- Over $2n+1$ network uses, with precoding,

$$\underline{Y_j} = \sum_{i=1}^{3} \underbrace{\begin{bmatrix} M_{ij}(\underline{\epsilon}_1) & 0 & \cdots & 0 \\ 0 & M_{ij}(\underline{\epsilon}_2) & \cdots & 0 \\ 0 & 0 & \vdots & 0 \\ 0 & 0 & \cdots & M_{ij}(\underline{\epsilon}_{2n+1}) \end{bmatrix}}_{H_{ij}} V_i \underline{X_i}$$

[KoM] R. Koetter, M. Medard, "An Algebraic Approach to Network Coding", IEEE/ACM Transactions on Networking, Vol. 11, No. 5, pp. 782-795, Oct. 2003.

# IA for 3-Unicast Session vs IA for $K$-user GIC

| 3-Unicast | 3-User GIC |
| --- | --- |
| Channel matrices and symbols from finite field | Channel matrices and symbols from complex field |
| Human control of the channel | Nature's control on the channel |
| Interdependency of channels | Independence of channels |
| Feasibility depends on topology | Feasibility in almost surely sense |

# IA for 3-Unicast Session

- Same choice of precoding matrices as for 3-User GIC.
- Conditions on the network topology under which (asymptotic) rates of $\frac{1}{2}$ is feasible - called "coupling relations".

# Feasibility Conditions for IA for 3-Unicast Session

- Define network polynomials
  - $p_1(\underline{\epsilon}) = \frac{M_{21}(\underline{\epsilon})M_{13}(\underline{\epsilon})}{M_{11}(\underline{\epsilon})M_{23}(\underline{\epsilon})}$,
  - $p_2(\underline{\epsilon}) = \frac{M_{22}(\underline{\epsilon})M_{13}(\underline{\epsilon})}{M_{12}(\underline{\epsilon})M_{23}(\underline{\epsilon})}$,
  - $p_3(\underline{\epsilon}) = \frac{M_{33}(\underline{\epsilon})M_{12}(\underline{\epsilon})}{M_{13}(\underline{\epsilon})M_{32}(\underline{\epsilon})}$,
  - $\eta = \frac{M_{32}(\underline{\epsilon})M_{21}(\underline{\epsilon})M_{13}(\underline{\epsilon})}{M_{12}(\underline{\epsilon})M_{31}(\underline{\epsilon})M_{23}(\underline{\epsilon})}$.

$$p_i = \frac{M_{ab}(\underline{\epsilon})M_{pq}(\underline{\epsilon})}{M_{aq}(\underline{\epsilon})M_{pb}(\underline{\epsilon})}$$

# Feasibility Conditions for IA for 3-Unicast Session (Contd.)

Case 1): $\eta \neq 1$.
Coupling Relation 1:

$$p_i \notin \left\{1, \eta, 1+\eta, \frac{\eta}{1+\eta}\right\}.$$

- Coupling relation invariant to choice of precoding matrices.

# Feasibility Conditions for IA for 3-Unicast Session (Contd.)

Case 2): $\eta = 1$.
Coupling Relation 2: Rate of $\frac{1}{2}$ iff $p_i \neq 1$.

- $V_1 = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $V_2 = M_{13}^{-1} M_{23} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$, $V_3 = M_{12}^{-1} M_{32} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$.
- Interpretation in terms of network topology and polynomial-time algorithms to check coupling relations in [MDRJMV].

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[MDRJMV] C. Meng et al., "Precoding-Based Network Alignment For Three Unicast Sessions", arXiv:1305.0868 [cs.IT].

# Outline

# Acyclic Networks with Delay

Why transform technique?

- No advantage in terms of complexity of decoding.
- Enables IA in 3-unicast sessions with delay without making using of memory at intermediate nodes.

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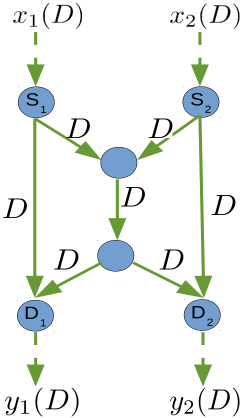
[BAPR] T. Bavirisetti, A. Ganesan, K. Prasad, and B. S. Rajan, "Precoding Based Network Alignment using Transform Approach for Acyclic Networks with Delay", arXiv:1310.2809 [cs.IT] (revised manuscript under review in IEEE Trans. Info. Theory).

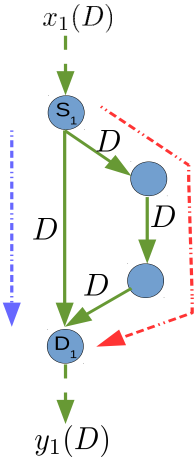
# Transfer Matrices in Acyclic Networks with Delay

$$x_i(D) = x_i^{(0)} D^0 + x_i^{(1)} D^1 + x_i^{(2)} D^2 + \cdots$$



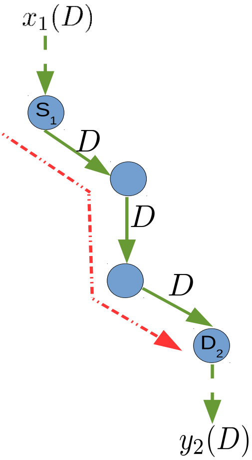

$$y_j(D) = y_j^{(0)} D^0 + y_j^{(1)} D^1 + y_j^{(2)} D^2 + \cdots$$

# Transfer Matrices in Acyclic Networks with Delay (Contd.)



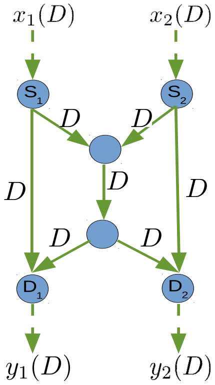

$$M_{11}(D) = D + D^3$$

# Transfer Matrices in Acyclic Networks with Delay (Contd.)




$$M_{12}(D) = D^3$$

# Transfer Matrices in Acyclic Networks with Delay (Contd.)




$$\begin{bmatrix} y_1(D) \\ y_2(D) \end{bmatrix} = \begin{bmatrix} D + D^3 & D^3 \\ D^3 & D + D^3 \end{bmatrix} \begin{bmatrix} x_1(D) \\ x_2(D) \end{bmatrix}$$

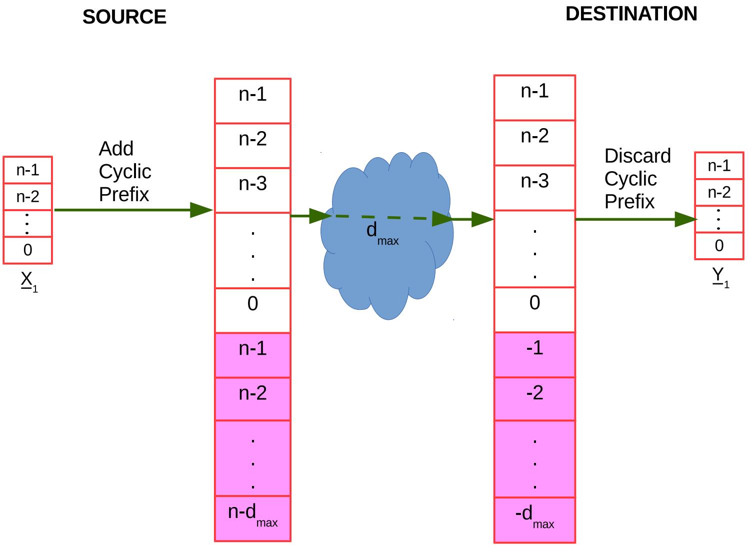
# Transfer Matrices in Acyclic Networks with Delay (Contd.)

- In general, $\underline{Y}(D) = M(D)\underline{X}(D)$, where [KoM]

$$M(D) = M^{(0)}D^0 + M^{(1)}D^1 + \cdots + M^{(d_{max})}D^{d_{max}},$$

  where $\{M^{(0)}, M^{(1)}, \cdots, M^{(d_{max})}\}$ is the impulse response of the network.
- Objective: To convert convolutional behaviour into instantaneous behaviour (inspired by OFDM for multipath wireless channels).

# Transform Technique



# Transform Technique (Contd.)

- Input-output relation after addition of CP and discarding the CP:

$$\begin{bmatrix} {Y_1}^{(n-1)} \\ {Y_1}^{(n-2)} \\ \vdots \\ {Y_1}^{(0)} \end{bmatrix} = \underbrace{\text{circ}(M_{11}^{(0)},\ M_{11}^{(1)}, \cdots,\ M_{11}^{(d_{max}-1)},\ M_{11}^{(d_{max})})}_{M_{11}} \begin{bmatrix} {X_1}^{(n-1)} \\ {X_1}^{(n-2)} \\ \vdots \\ {X_1}^{(0)} \end{bmatrix}.$$

# Transform Technique (Contd.)

$$M_{11} = \begin{bmatrix} M_{11}^{(0)} & M_{11}^{(1)} & \cdots & M_{11}^{(d_{max})} & 0 & \cdots & 0 & 0 \\ 0 & M_{11}^{(0)} & \cdots & M_{11}^{(d_{max}-1)} & M_{11}^{(d_{max})} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ M_{11}^{(1)} & M_{11}^{(2)} & \cdots & M_{11}^{(d_{max})}0 & 0 & \cdots & 0 & M_{11}^{(0)} \end{bmatrix}$$

# Transform Technique (Contd.)

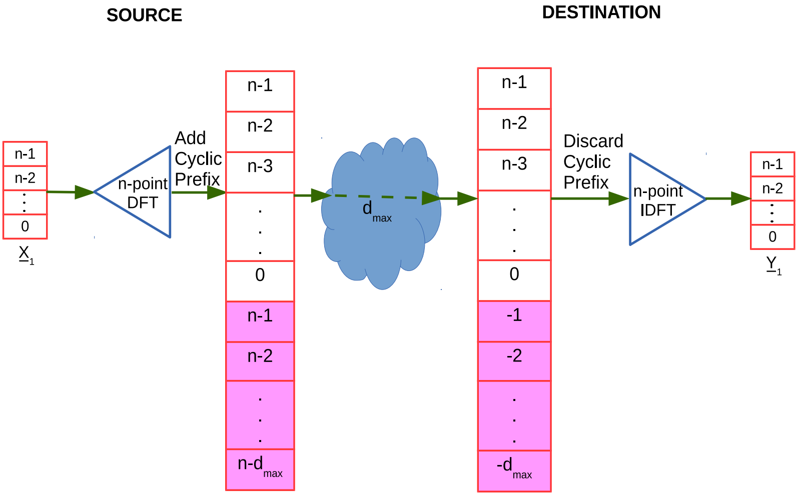
RESULT:
Circulant matrix $M_{11}$ can be diagonalized using finite-field DFT matrix given by

$$F = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \cdots & \alpha^{(n-1)(n-1)} \end{bmatrix},$$

where $n | q-1$, $\mathbb{F}_q$ being the field of operation. That is,

$$\hat{M}_{11} = F M_{11} F^{-1}$$

# Transform Technique (Contd.)



# Transform Technique (Contd.)

- Input-output relation is now given by

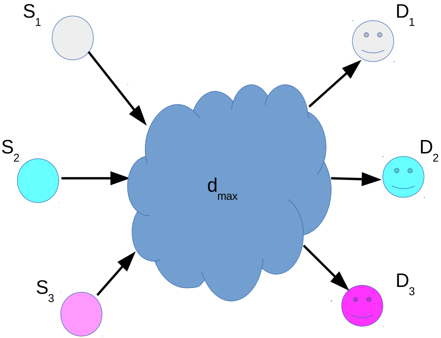
$$\begin{bmatrix} {Y_1}^{(n-1)} \\ {Y_1}^{(n-2)} \\ \vdots \\ {Y_1}^{(0)} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{d_{max}} M_{11}^{(i)} & 0 & 0 & \cdots & 0 \\ 0 & \sum_{i=0}^{d_{max}} \alpha^{i} M_{11}^{(i)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=0}^{d_{max}} \alpha^{(n-1)i} M_{11}^{(i)} \end{bmatrix} \begin{bmatrix} {X_1}^{(n-1)} \\ {X_1}^{(n-2)} \\ \vdots \\ {X_1}^{(0)} \end{bmatrix}.$$

  We now say that the single source single sink network has been transformed into *n*-instantaneous networks!
- No memory is used at the intermediate nodes.

# Outline

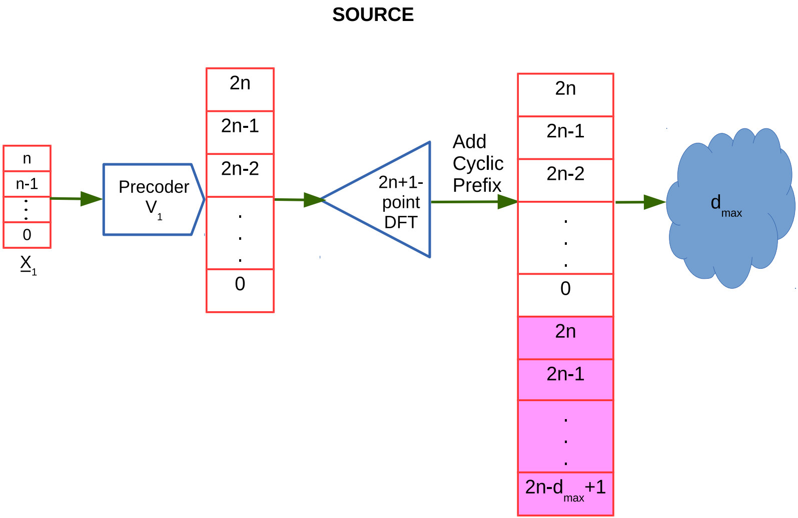
# IA with Time-Invariant LECs

# Three Unicast Sessions with Delay




$$Y_j = \hat{M}_{1j}X_1 + \hat{M}_{2j}X_2 + \hat{M}_{3j}X_3$$

# IA with Time-Invariant LECs



# IA with Time-Invariant LECs (Contd.))

$$Y_j = \hat{M}_{1j}(\underline{\epsilon})V_1X_1 + \hat{M}_{2j}(\underline{\epsilon})V_2X_2 + \hat{M}_{3j}(\underline{\epsilon})V_3X_3$$

RESULT:

The input symbols can be exactly recovered at destinations subject to $p \nmid 2n+1$, if the following conditions hold.

$$\text{Rank}[V_1 \quad \hat{M}_{11}^{-1}\hat{M}_{21}V_2] = 2n+1,$$
$$\text{Rank}[\hat{M}_{12}^{-1}\hat{M}_{22}V_2 \quad V_1] = 2n+1,$$
$$\text{Rank}[\hat{M}_{13}^{-1}\hat{M}_{33}V_3 \quad V_1] = 2n+1.$$

# Example - IA with Time-Invariant LECs

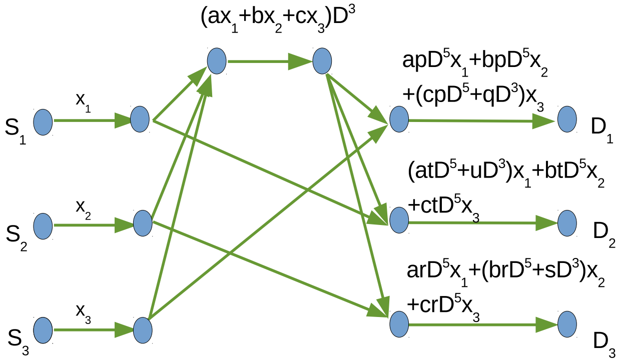



Figure : In this network, $d_{max} = 2$. IA is feasible over $GF(2^6)$ with $2n + 1 = 7$.

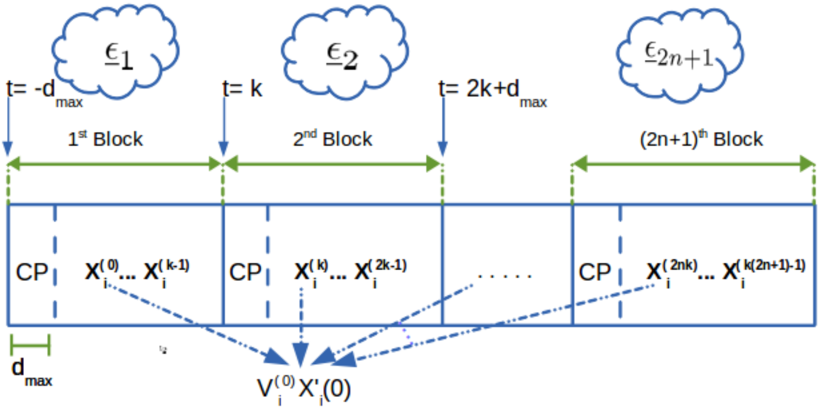
# IA with Time-Invariant LECs (Contd.)

Bottomline:

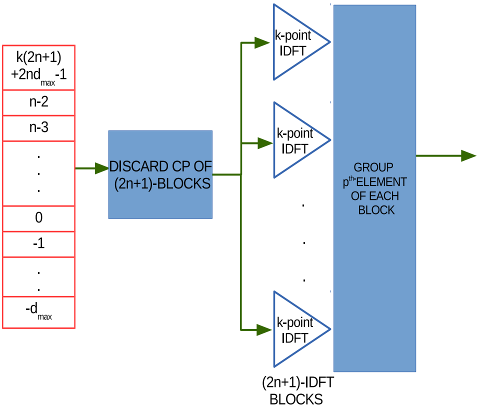
- Varying coding coefficients w.r.t is a necessity in instantaneous networks to achieve IA.
- Channel diversity required for IA in delay networks is supplied by delays. So, varying coding coefficients is not a necessity.

# IA with Block Time-Varying LECs

# IA with Block Time-Varying LECs



# IA with Block Time-Varying LECs (Contd.)




For $p = 0, 1, \cdots, k-1$

$$
\begin{aligned}
\underline{Y}_1^{(p)} = & \begin{bmatrix} M_{11}(\underline{\epsilon}_1, \alpha^p) & 0 & 0 & \cdots & 0 \\ 0 & M_{11}(\underline{\epsilon}_2, \alpha^p) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{11}(\underline{\epsilon}_{2n+1}, \alpha^p) \end{bmatrix} V_1^{(p)} X_1 \\
+ & \begin{bmatrix} M_{21}(\underline{\epsilon}_1, \alpha^p) & 0 & 0 & \cdots & 0 \\ 0 & M_{21}(\underline{\epsilon}_2, \alpha^p) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{21}(\underline{\epsilon}_{2n+1}, \alpha^p) \end{bmatrix} V_2^{(p)} X_2 \\
+ & \begin{bmatrix} M_{31}(\underline{\epsilon}_1, \alpha^p) & 0 & 0 & \cdots & 0 \\ 0 & M_{31}(\underline{\epsilon}_2, \alpha^p) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{31}(\underline{\epsilon}_{2n+1}, \alpha^p) \end{bmatrix} V_3^{(p)} X_3
\end{aligned}
$$

# Feasibility of IA with Block Time-Varying LECs

RESULT: Necessary and Sufficient conditions for feasibility of IA using Block Time Varying LECs is exactly the same as "coupling relations" for the instantaneous network counterpart derived in [MRMJ].

- No need to simulate instantaneous network behaviour in 3-unicast sessions with delay.

# IA with Time-Varying LECs

# Generalizing IA with Time-Varying LECs

- Rate($S_i - D_i$) $= \frac{n_i}{n}$.
- Vary LECs with every time-instant.
- Addition and discarding of CP, gives an input-output relation

$$\underline{Y}_j = \sum_{i=1}^{3} M_{ij} V_i \underline{X_i}.$$

# Feasibility of IA with Time-Varying LECs

Is IA with time-varying LECs feasible, for some $(n_i > 0, n > 0)$, when the other two IA schemes fail in 3-unicast sessions with delay?

# IA with Block Time-Varying LECs (Contd.)

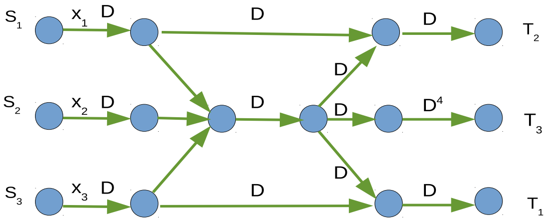



Figure : IA with Time-Varying LECs is feasbile for $n_1 = 5, n_2 = 3, n_3 = 3, n = 8$. IA with Time-Invariant and Block Time-Varying LECs is infeasible.

# Outline

# Summary

- Transform technique for acyclic networks with delays was discussed.
- Three different IA schemes for 3-unicast sessions with delays and their relation with IA for 3-unicast instantaneous networks were discussed

# Open Problems

- Insights into networks where IA with time-varying LECs is feasible.
- Extensions of IA to higher dimensional networks.

THANK YOU!